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„Local geometry of the space of arcs“

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Introduction

Let X be a variety over a field k. An *arc* on X over k is a morphism of schemes $\alpha : \mathrm{Spec}(k[[t]]) \to X$. If X is defined by polynomials $f_1, \ldots, f_r \in k[x_1, \ldots, x_N]$ inside $\mathbb{A}_k^N$, then α corresponds to a solution of the system

$$\begin{aligned} f_1(x_1(t), \ldots, x_N(t)) &= 0 \\ &\cdots \\ f_r(x_1(t), \ldots, x_N(t)) &= 0 \end{aligned} \tag{$\star$}$$

in $k[[t]]^N$. If $k = \mathbb{C}$ and the series $x_1(t), \ldots, x_N(t)$ are convergent, then one may think of α as a parametrization of a germ of an analytic curve living on the complex variety X. Similarly, for $n \in \mathbb{N}$, an *n-jet* on X over k is a morphism $\alpha_n : \mathrm{Spec}(k[t]/(t^{n+1})) \to X$, or, equivalently, a solution to ($\star$) in $(k[t]/(t^{n+1}))^N$. A 1-jet is nothing but a tangent vector to X; higher order jets may be thought of as higher order infinitesimal information of X. Similar concepts are ubiquitous throughout many different fields of mathematics, the jet bundle associated to a fiber bundle in differential topology being just one example.

For $n \in \mathbb{N}$ the *n-th jet space* X_n of X is the scheme parametrizing jets of order n on X. More precisely, X_n is the scheme representing the n-th jet functor

$$Y \in \mathrm{Sch}_k \mapsto \mathrm{Hom}_k(Y \times_k \mathrm{Spec}(k[t]/(t^{n+1})), X).$$

For $m > n$, the natural maps $k[t]/(t^{m+1}) \to k[t]/(t^{n+1})$ induce *truncation morphisms* $\pi_{m,n} : X_m \to X_n$. The *arc space* X_∞ of X is defined to be the projective limit $X_\infty = \lim_n X_n$, which is again a scheme. In particular, X_∞ satisfies

$$\mathrm{Hom}_k(\mathrm{Spec}(k), X_\infty) \simeq \mathrm{Hom}_k(\mathrm{Spec}(k[[t]]), X),$$

i.e. it parametrizes arcs on X. If X is positive-dimensional, then X_∞ is a non-Noetherian scheme of infinite dimension. The study of the geometry of X_n and X_∞ will be the main subject of this monograph. We assume the reader to be familiar with commutative algebra and algebraic geometry at the level of Grothendieck's EGA. While we will introduce all the necessary definitions and terminology related to the theory of arc spaces in Chapter 3, for a more comprehensive introduction we refer to the many surveys throughout the literature, such as [Vey06; Mus07; Ish12; dFe18].

Let us first give a brief and by no means comprehensive overview of the history of the study of jet and arc spaces and their geometry. One of the first major works was [Nas95]. Written by Nash in 1968, a preprint was in circulation soon thereafter, with the cited version published only much later in

1995. Motivated by Hironaka's then-recent proof of resolution of singularities in characteristic 0, Nash conjectured a deep relation between certain components of the arc space X_∞ of a variety X and the *essential divisors* of X, which are exceptional divisors appearing on every resolution of X. More precisely, let $\operatorname{Sing} X$ denotes the singular locus of X and $\pi : X_\infty \to X$ the natural projection. Nash proved that the spce of arcs centered at the singular locus $\pi^{-1}(\operatorname{Sing} X)$ has only finitely many irreducible components and, moreover, that by lifting resolutions of singularities of X to X_∞, one obtains an injective map

$$\{\text{irreducible components of } \pi^{-1}(\operatorname{Sing} X)\} \to \{\text{essential divisors of } X\}. \quad (\star\star)$$

This map can be extended to positive characteristics by only considering those components of $\pi^{-1}(\operatorname{Sing} X)$ which are not contained in $(\operatorname{Sing} X)_\infty$. Arcs not contained in $\operatorname{Sing} X$ are often called *non-degenerate*.

When X is a surface, its essential divisors correspond to the exceptional components of the minimal resolution $X' \to X$. In that case, Nash asked whether the map $(\star\star)$ is a bijection; this question became known as the *Nash problem*. After many partial results for various classes of surface singularities, the Nash problem was fully solved only in 2012 by Bobadilla and Pe Pereira in [BP12] (over algebraically closed fields of characteristic 0). Their proof is topological in nature and uses lifting of convergent wedges, which should be thought of as uniparametric families of arcs, together with the topological Euler characteristic. A more algebraic proof was given later in [dFD16] from the point of view of the minimal model program.

In higher dimensions the above map fails to be bijective in general; explicit counterexamples were given in every dimension ≥ 3 (see for example [dF13]). Nevertheless, there are several classes of varieties for which the map was verified to be a bijection in all dimensions, for example toric varieties, see [IK03].

The space of arcs was also implicitly considered in the works of Kolchin et. al. on differential algebra (see for example [Kol73]). In characteristic 0, a way of constructing the arc space of a variety X is to consider the closure of the defining ideal $I(X)$ under some derivation D. In [Rib71a; Rib71b] and later [Voj07] the positive characteristic case was treated by considering higher order derivations instead. In particular, if $X = \operatorname{Spec}(A)$ is affine, then $X_n = \operatorname{Spec}(\mathbb{HS}^n_k(A))$, where $\mathbb{HS}^n_k(A)$ is the universal algebra of higher derivations of order n, which we call *Hasse–Schmidt algebra*. We will review this approach in Sections 2.1 and 2.2.

Originally introduced in a talk by Kontsevich in 1995 and further developed by Batyrev in [Bat98; Bat99] and Denef and Loeser in [DL99a; DL99b], motivic integration is based on the idea of replacing the p-adic measure used in p-adic integration with the so-called *motivic measure*, which is defined on constructible subsets of X_∞ and classically takes values in the completion $\widehat{\mathcal{M}}_k$ of the Grothendieck group of varieties. Crucial to some of the most important applications of motivic integration was the study of the geometric properties of jet and arc spaces. Of particular interest here is the following result in [DL99a] (also referred to as the *Denef–Loeser fibration lemma*). In the context of this monograph, this result should be understood as a regularity condition for non-degenerate arcs.

Theorem ([DL99a, Lemma 4.1])**.** *Let X be an algebraic variety over an algebraically closed field k which is equidimensional of dimension d. Denote by $X^{(e)}_\infty$*

the subset of arcs having contact order $\leq e$ *with* $\operatorname{Sing} X$ *and by* $\pi_n : X_\infty \to X_n$ *the natural projections. Then, for* $n \gg 0$*, the map*

$$\pi_{n+1}(X_\infty) \to \pi_n(X_\infty)$$

is a piecewise trivial fibration over $\pi_n(X_\infty^{(e)})$ *with fiber* $\mathbb{A}_k^d$*.*

Another important result proven in [DL99a] is a change of variables formula for motivic integrals. Implicitly considered in this formula is what is now referred to as *Mather discrepancy* $\widehat{k}_E$ for a divisor E on a (singular) variety X. It is defined as $\widehat{k}_E = \operatorname{ord}_E(\operatorname{Jac}_f)$, where Jac_f is the Jacobian ideal associated to a resolution of singularities $f : Y \to X$. The Mather discrepancy as an invariant of singularities and its connection to the space of arcs were studied in, among others, [Ish13; dFEI08]. It is one example of an invariant obtained from *divisorial valuations*, which themselves are closely related to the geometry of jet and arc spaces, as explored for example in [Mus01; Mus02; ELM04; Ish08].

In contrast to the purely topological properties of arc spaces, the study of their schematic structure is still a relatively recent development. One of the first major results in that direction is the theorem of Drinfeld, Grinberg and Kazhdan.

Theorem ([GK00; Dri02])**.** *Let* X *be a scheme of finite type over a field* k *and* $\alpha \in X_\infty(k)$ *a non-degenerate arc. Then we have an isomorphism of formal neighborhoods*

$$\widehat{(X_\infty)_\alpha} \simeq \widehat{Z}_z \times \Delta^{\mathbb{N}},$$

where Z *is a scheme of finite type over* k *and* $\Delta^{\mathbb{N}} = \operatorname{Spf}(k[[t_n \mid n \in \mathbb{N}]])$ *(with* Spf *denoting the formal spectrum).*

One should note that this result is not a direct consequence of the Denef–Loeser fibration lemma. The Drinfeld–Grinberg–Kazhdan theorem says that the singularities of non-degenerate arcs are encoded in the finite-dimensional scheme Z, which is sometimes referred to as a *formal model* of α. Formal models of arcs on X and their relation to the singularities of X itself were studied by Bourqui and Sebag in [BS17c; BS17b; BS17a; BS17d; BS19b], though this relation is still far from being understood in its entirety.

Several attempts were also made to obtain a more global version of the Drinfeld–Grinberg–Kazhdan theorem, most notably in [BNS16] and [Ngo17]. The main motivation in there is the expectation that there should exist a notion of intersection complex on the space of arcs, which itself should be related to the singularities of other infinite-dimensional spaces appearing in, among others, the Geometric Langlands Program. A precise connection was established in [BNS16] by defining a function on the stalks of the intersection complex of the formal models of non-degenerate arcs.

Variants of the Drinfeld–Grinberg–Kazhdan theorem for spaces of formal power series have also appeared in [BH10; HW]. We should also mention the work of Reguera in [Reg06; Reg09] on the formal neighborhood of *stable points*, which are the generic points of irreducible constructible subsets of the arc space. In [MR18] it was shown that the embedding dimension of X_∞ at stable points corresponding to some divisorial valuation ν_E is related to the Mather discrepancy $\widehat{k}_E$, while the dimension of the local ring is bounded from below by the

Mather–Jacobian discrepancy with respect to E. We will review this result in Section 3.6.

The main goal of this monograph is to study the local *schematic* structure of the space of arcs, hence in particular its singularities as a (non-reduced) scheme. As the arc space is non-Noetherian in all interesting cases, there does not exist a clear substitute for the étale topology[1] and hence we will mainly study the formal neighborhoods of arcs as in the Drinfeld–Grinberg–Kazhdan theorem. Let us give a brief summary of each chapter.

One of the challenges of studying the formal neighborhoods of arcs $\alpha \in X_\infty$ is that the underlying rings $\widehat{\mathcal{O}_{X_\infty,\alpha}}$ elude many of the common methods and tools of commutative algebra. The main goal of Chapter 1 is thus to describe these rings and provide suitable extensions of classical results. In Section 1.1 we start by reviewing not necessarily adic formal schemes, with the main example being formal completions of non-Noetherian k-schemes. We compare it with some of the slightly differing definitions of formal schemes in the literature.

In Section 1.2, we first show that the natural topology on the formal power series ring $\widehat{P} := k[[t_i \mid i \in I]]$ is not adic if $|I| = \infty$. Indeed, if $\widehat{\mathfrak{m}}$ denotes the maximal ideal of $\widehat{P}$, then a neighborhood basis for 0 in $\widehat{P}$ is given by a descending filtration

$$\widehat{\mathfrak{m}} = \widehat{\mathfrak{m}}_1 \supset \widehat{\mathfrak{m}}_2 \supset \widehat{\mathfrak{m}}_3 \supset \dots,$$

strictly coarser than the usual $\widehat{\mathfrak{m}}$-adic filtration. However, this filtration is approximated by the adic filtration, in a sense we make precise in Section 1.3. There we introduce the notion of *quasi-adic* rings as admissible topological rings A such that, for every ideal of definition $\mathfrak{a} \subset A$, the quotient $A/\mathfrak{a}$ is adic. We then prove an extension of the classical Cohen structure theorem as follows.

Theorem A (1.3.35)**.** *Let $(A, \mathfrak{m}, K)$ be a local quasi-adic ring which is equicharacteristic and let k be the prime field contained in A. Then there exists a* efficient formal embedding f *of A, that is, a surjective map of topological rings*

$$f : K[[t_i \mid i \in I]] \to A$$

inducing an isomorphism on continuous cotangent spaces (see Definition 1.3.25). Moreover, an efficient formal embedding f of A is an isomorphism if and only if A is formally smooth over k.

Note that here formal smoothness is used to replace the condition of A being a regular local ring as in the original Cohen structure theorem. We will also briefly cover the case of mixed characteristic and show that, as in the Noetherian case, there always exists a ring of coefficients for A.

It is a well-known fact that for any Noetherian complete local ring $(A, \mathfrak{m})$ its $\mathfrak{m}$-adic completion $\widehat{A}$ is flat over A. In Section 1.4 we will prove the same for the infinite-variate polynomial ring $P = k[t_i \mid i \in I]$ and its completion, $\widehat{P} = k[[t_i \mid i \in I]]$. However, we will also give an example of a separated local k-algebra A whose adic completion $\widehat{A}$ is not flat over A, showing that flatness of the completion map is rather delicate in the non-Noetherian case.

In Section 1.5 we extend the Grauert–Hironaka division theorem as well as the Buchberger criterion for standard bases to infinite-variate formal power

[1] For an in-depth discussion see [Ngo17].

series. Note that we will always require the existence of a finite standard basis. In that case, the proofs are straightforward, though some care has to be taken when considering different monomial orders. We shall mention here that it is not clear for which ideals of $\widehat{P}$ there exist a finite standard basis.

Finally, in Section 1.6 we consider ideals $\mathfrak{a}$ of $\widehat{P}$ which are generated by finitely many elements $f_1, \ldots, f_n \in \mathfrak{a}$, which themselves only involve finitely many variables of $\widehat{P}$. We call such $\mathfrak{a}$ *ideals of finite definition* and prove the following comparison theorem.

Theorem B (1.6.6, 1.6.9)**.** *Let $J \subset I$ be a finite subset and consider $\widehat{P}_J := k[[x_j \mid j \in J]]$ as a (Noetherian) subring of $\widehat{P} = k[[x_i \mid i \in I]]$. Let $f_1, \ldots, f_r \in \widehat{P}_J$ and $\mathfrak{a}_J$, $\mathfrak{a}$ be the ideals of $\widehat{P}_J$, $\widehat{P}$ generated by them. Then every minimal prime over $\mathfrak{a}$ is of the form $\mathfrak{p}\widehat{P}$, where $\mathfrak{p}$ is a minimal prime over $\mathfrak{a}_J$. Moreover, we have*

$$\mathrm{ht}(\mathfrak{a}) = \mathrm{ht}(\mathfrak{a}_J).$$

In particular, if $\mathfrak{a} \subset \widehat{P}$ is of finite definition, then $\mathrm{ht}(\mathfrak{a}) < \infty$. The proof of this theorem relies both on an extension of the Weierstrass preparation theorem proven in Section 1.2 as well as the flatness results established in Section 1.4.

The schematic structure of the arc space X_∞ of a k-scheme X was studied extensively in [dFD20] by means of the sheaf of differentials $\Omega_{X_\infty/k}$. The main result there is a formula relating $\Omega_{X_\infty/k}$ to the sheaf of differentials on the original variety $\Omega_{X/k}$. In the affine case, the formula reads as follows:

Theorem ([dFD20, Theorem 5.3])**.** *Let A be a k-algebra and, for $n \in \mathbb{N} \cup \{\infty\}$, let $A_n := \mathbb{HS}^n_k(A)$ be the n-th* Hasse–Schmidt algebra *of A (see Section 2.1). Then*

$$\Omega_{A_n/k} \simeq \Omega_{A/k} \otimes_A Q_n,$$

where Q_n is the A_n-dual of $A_n[t]/(t^{n+1})$ in the case $n \in \mathbb{N}$.

As the n-th Hasse–Schmidt algebra $\mathbb{HS}^n_k(A)$ parametrizes infinitesimal data on A up to order n (i.e. jets of order n on $\mathrm{Spec}(A)$), this formula suggests that tangents (i.e. infinitesimal data up to order 1) of n-jets on A can be recovered from some higher order operation on tangents on A. We will make this idea precise by considering higher order derivations of modules, as introduced by Ribenboim in [Rib80]. As for higher derivations of rings, there too exists a universal object parametrizing such module derivations, which we call the *Hasse–Schmidt module*. The study of both Hasse–Schmidt algebras and Hasse–Schmidt modules will be the main goal of Sections 2.1 and 2.2. In particular, we will prove the following comparison theorem:

Theorem C (2.2.12)**.** *Let A be a k-algebra and M an A-module. Then, for $n \in \mathbb{N} \cup \{\infty\}$, there exists a natural isomorphism*

$$\mathrm{Sym}_{A_n}(\mathbb{HS}^n_{A/k}(M)) \simeq \mathbb{HS}^n_k(\mathrm{Sym}_A(M)),$$

where $A_n = \mathbb{HS}^n_k(A)$ is the n-th Hasse–Schmidt algebra of A and $\mathbb{HS}^n_{A/k}(M)$ is the n-th Hasse–Schmidt module of M.

As a consequence, we obtain the above formula for $\Omega_{A_n/k}$ via the elementary observation that the Hasse–Schmidt algebra functors $\mathbb{HS}^n_k(-)$ commute.

In Section 2.3 we will prove an extension of the classical Zariski–Lipman–Nagata criterion for non-Noetherian quasi-adic rings, as done in [CH17; BS19a]. Roughly speaking, it gives a criterion for the existence of *smooth factors*, that is, isomorphisms of the form $A \simeq A' \widehat{\otimes}_k k[[t]]$, in terms of the existence of continuous regular higher derivations. As a consequence we get the following cancellation result:

Theorem D ([BS19a, Theorem 1.2]). *Let $(A, \mathfrak{m})$, $(A', \mathfrak{m}')$ be two local equicharacteristic quasi-adic rings with $A/\mathfrak{m} = A'/\mathfrak{m}' = k$ and such that A and A' have no smooth factors. Assume there exists an isomorphism*

$$\varphi : A \widehat{\otimes}_k k[[u_i \mid i \in I]] \simeq A' \widehat{\otimes}_k k[[v_j \mid j \in J]]$$

of local quasi-adic rings. Then φ induces an isomorphism

$$\varphi' : A \simeq A'.$$

By the Drinfeld–Grinberg–Kazhdan theorem, this theorem in particular asserts the existence of *minimal* formal models of non-degenerate arcs.

Although we mostly aim to apply the foundational results of Chapters 1 and 2 to the singularities of arc spaces, we will make a brief digression to the case of algebraic varieties in Section 2.4. There we first review the use of the minimal formal model as an invariant of singularities by showing that taking its dimension gives rise to a upper semi-continuous function on the variety X. We will then extend a result originally proven by [Eph78] in the case of analytic spaces to the algebraic setting as follows.

Theorem E (2.4.10, 2.4.14, 2.4.16). *Let X be a scheme of finite type over an algebraically closed field k. For each $p \in X(k)$ consider the set*

$$\operatorname{Iso}(X, p) := \{q \in X(k) \mid \widehat{X}_p \simeq \widehat{X}_q\},$$

called the isosingular locus *of p. Then*

1. *For every $p \in X(k)$ the set $\operatorname{Iso}(X, p)$ is locally closed in $X(k)$. Let $X^{(p)}$ denote the unique reduced subscheme of X whose k-points equal $\operatorname{Iso}(X, p)$. Then $X^{(p)}$ is smooth.*

2. *If in addition $\operatorname{char}(k) = 0$, then for every $p \in X(k)$ there exists an isomorphism of formal schemes*

$$\widehat{X}_p \simeq \widehat{(X^{(p)})}_p \times \widehat{Y}_y,$$

which is maximal, that is, $\widehat{Y}_y$ has no smooth factors.

Thus, in the case of varieties, the minimal formal model of $p \in X(k)$ can be interpreted as a transversal slice to the isosingular locus of p. As the proof of the theorem requires the use of the *contact group* and a variant of Artin approximation (c.f. [Art69]) to reduce to the case of algebraic groups, it is not clear how to extend it to the non-Noetherian case.

In Section 2.5 we consider the *embedding codimension* as another formal invariant of singularities. If p is a point of a variety X, then the embedding codimension (sometimes also referred to as *regularity defect*, as in [Lec64]) of its local ring $\mathcal{O}_{X,p}$ is simply given by the difference

$$\mathrm{ecodim}(\mathcal{O}_{X,p}) = \mathrm{edim}(\mathcal{O}_{X,p}) - \dim_p(X).$$

In particular $p \in X$ is regular if and only if $\mathrm{ecodim}(\mathcal{O}_{X,p}) = 0$. We give two ways of extending this definition to arbitrary schemes X. The first one uses the embedding of the tangent cone in the Zariski tangent space. More precisely, if $p \in X$ and $\mathfrak{m}$ is the maximal ideal of the local ring $\mathcal{O}_{X,p}$, then there exists a natural surjection

$$\gamma : \mathrm{Sym}(\mathfrak{m}/\mathfrak{m}^2) \to \mathrm{gr}(\mathcal{O}_{X,p}).$$

We then set

$$\mathrm{ecodim}(\mathcal{O}_{X,p}) := \mathrm{ht}(\ker(\gamma)).$$

In the case where X is equicharacteristic, that is, X is a scheme over a field k, we provide an alternative way of defining embedding codimension by means of the formal completion of X. Using the extended Cohen structure theorem established in Section 1.3, there exists a surjection of quasi-adic rings

$$\alpha : K[[t_i \mid i \in I]] \to \widehat{\mathcal{O}_{X,p}},$$

which induces an isomorphism of continuous cotangent spaces. Then we define the *formal embedding codimension* as

$$\mathrm{fcodim}(\mathcal{O}_{X,p}) := \mathrm{ht}(\ker(\alpha)).$$

Using the degeneration of $\mathcal{O}_{X,p}$ to its extended Rees algebra we obtain the following comparison result:

Theorem F (2.5.19)**.** *Let X be a scheme over a field k and $p \in X$. Then*

$$\mathrm{ecodim}(\mathcal{O}_{X,p}) \leq \mathrm{fcodim}(\mathcal{O}_{X,p}).$$

The embedding codimension will be one of the main tools used in Chapter 3, which will be devoted to the study of the (formal) local geometry of arc spaces. We start by reviewing the basic definitions and results in Section 3.1. In Section 3.2, we will first summarize some of the results obtained in [dFD20]. We show that by glueing the corresponding Hasse–Schmidt modules, one obtains a functor associating to any coherent sheaf $\mathcal{F}$ on X a coherent sheaf $\mathcal{F}_n$ on X_n resp. a quasi-coherent sheaf $\mathcal{F}_\infty$ on X_∞. In particular, we recover the full formula for the sheaf of differentials of X_∞. As one of the consequences obtained in [dFD20] we should mention here Lemma 3.2.10, which says that, for $n \in \mathbb{N}$ and α_n the truncation of an arc α,

$$\mathrm{edim}(\mathcal{O}_{X_n,\alpha_n}) = (n+1)d_n + \mathrm{ord}_\alpha(\mathrm{Fitt}^{d_n}(\Omega_{X/k})).$$

The integer d_n here is an invariant associated to the pullback of $\Omega_{X/k}$ over α_n. If the arc α is non-degenerate, then $d_n = \dim_{\alpha_\eta} X$ for $n \gg 0$; otherwise $d_n > \dim_{\alpha_\eta} X$ for all $n \in \mathbb{N}$.

At the end of Section 3.2 we will present the main result of [CH17]:

Theorem G (3.2.14)**.** *Let X be a scheme of finite type over a field k of characteristic 0. Let $p \in X(k)$ and, for $n \in \mathbb{N} \cup \{\infty\}$, let α_p be the constant n-jet (respectively arc) centered in p. Then the formal neighborhood $\widehat{(X_n)}_{\alpha_p}$ has a smooth factor if and only if $\widehat{X}_p$ does.*

We prove this theorem here by combining the Zariski–Lipman–Nagata criterion with the formula for the sheaf of differentials of X_n in [dFD20]. As a particular consequence we get that a constant arc centered in a singular point of X does not have a decomposition as in the Drinfeld–Grinberg–Kazhdan theorem. This was first verified for a specific example in [BS17c].

Section 3.3 is devoted to proving the following characterization of non-degenerate arcs:

Theorem H (3.3.5, 3.3.7)**.** *Let X be a scheme of finite type over a field k and $\alpha \in X_\infty$. Assume either $\operatorname{char}(k) = 0$ or $\alpha \in X_\infty(k)$. Then the following are equivalent:*

1. *α is non-degenerate, i.e. $\alpha \in X_{\mathrm{sm}}$.*
2. *$\operatorname{ecodim}(\mathcal{O}_{X_\infty,\alpha}) < \infty$.*
3. *$\operatorname{ecodim}(\mathcal{O}_{X_\infty,\alpha}) < \operatorname{ord}_\alpha(\operatorname{Fitt}^d(\Omega_{X/k}))$, where $d = \dim_{\alpha_\eta}(X)$.*

In particular, if X is a variety, then (3) is equivalent to

$$\operatorname{ecodim}(\mathcal{O}_{X_\infty,\alpha}) \leq \operatorname{ord}_\alpha(\operatorname{Jac}_X).$$

Let us briefly sketch the main ideas and methods used in the proof of the theorem. The first step is the following crucial observation:

Theorem I (3.3.1)**.** *Let X, α be as before and let $f : X \to Y = \mathbb{A}^d_k$ be a general linear projection, with $d = \dim_{\alpha_\eta}(X)$. Let $\mathfrak{m} \subset \mathcal{O}_{X_\infty,\alpha}$ and $\mathfrak{n} \subset \mathcal{O}_{Y_\infty,\beta}$ be the respective maximal ideals and L and L' the residue fields. Then the cotangent map induced by f*

$$(T_\alpha f_\infty)^* \colon \mathfrak{n}/\mathfrak{n}^2 \otimes_{L'} L \to \mathfrak{m}/\mathfrak{m}^2$$

is an isomorphism.

In particular, the induced surjection on associated graded rings

$$k[t_i \mid i \in \mathbb{N}] \simeq \operatorname{gr}(\mathcal{O}_{Y_\infty,\beta}) \to \operatorname{gr}(\mathcal{O}_{X_\infty,\alpha})$$

can be used to compute $\operatorname{ecodim}(\mathcal{O}_{X_\infty,\alpha})$. We take advantage of the fact that this map is the colimit of the corresponding maps at the level of jets and, for $m \leq n$, consider the tower:

$$\begin{tikzcd}
\operatorname{gr}(\mathcal{O}_{Y_\infty,\beta}) \otimes_{L'} L \arrow[r, "\varphi"] & \operatorname{gr}(\mathcal{O}_{X_\infty,\alpha}) \\
\operatorname{gr}(\mathcal{O}_{Y_n,\beta_n}) \otimes_{L'_n} L \arrow[r, "\varphi_n"] \arrow[u] & \operatorname{gr}(\mathcal{O}_{X_n,\alpha_n}) \otimes_{L_n} L \arrow[u] \\
\operatorname{gr}(\mathcal{O}_{Y,\beta_m}) \otimes_{L'_m} L \arrow[r, "\varphi_m"] \arrow[u] & \operatorname{gr}(\mathcal{O}_{X_m,\alpha_m}) \otimes_{L_m} L \arrow[u]
\end{tikzcd}$$

Note that $\mathrm{gr}(\mathcal{O}_{Y_n,\beta_n})$ is a polynomial ring of dimension $d(n+1)$. Together with the previously mentioned formula for the embedding dimension of $\mathcal{O}_{X_n,\alpha_n}$ we obtain

$$\mathrm{ecodim}(\mathcal{O}_{X_\infty,\alpha}) = \mathrm{ht}(\ker(\varphi)) = \limsup_n \mathrm{ht}(\ker(\varphi_n)) \leq \mathrm{ord}_\alpha(\mathrm{Fitt}^d(\Omega_{X/k})).$$

This shows that (1) implies (3) in the above theorem. The converse direction can be proven in a similar way.

In Section 3.4 we further discuss this result in conjunction with the Drinfeld–Grinberg–Kazhdan theorem. We see that the full converse to the latter holds: namely, any arc $\alpha \in X_\infty(k)$ whose formal neighborhood decomposes as

$$\widehat{(X_\infty)}_\alpha \simeq \widehat{Z}_z \times \mathrm{Spf}(k[[t_i \mid i \in \mathbb{N}]]),$$

with Z of finite type over k, is already non-degenerate. Moreover, we see that the formal embedding induced by a general projection is efficient and, after a change of formal coordinates, yields an efficient embedding of the *minimal* formal model. We should note that the explicit construction given by Drinfeld's proof in [Dri02; Dri18] does not give the minimal formal model in general. We also provide an explicit example which proves that the bound for the embedding codimension established in (3) of Theorem H is sharp.

One thing to note about the decomposition established in the Drinfeld–Grinberg–Kazhdan theorem is that it does not arise as the completion of a map of (usual) schemes. This is partly due to the geometric properties of the Weierstrass preparation theorem, which we briefly discuss at the end of Section 3.4. On the other hand, the composition with the induced map obtained from a general projection $f : X \to Y = \mathbb{A}^d$ as

$$\widehat{Z}_z \times \mathrm{Spf}(k[[t_i \mid i \in \mathbb{N}]]) \simeq \widehat{(X_\infty)}_\alpha \to \widehat{(Y_\infty)}_\beta$$

does arise as the completion of a morphism $Z \times \mathbb{A}^{\mathbb{N}} \to Y$. By carefully analyzing tangent maps in Section 3.5 we see that the Drinfeld formal model $\widehat{Z}_z$ embeds into the formal neighborhood $\widehat{(Y_{2e-1})}_{\beta_{2e-1}}$ of the jet space of order $2e-1$ of $Y = \mathbb{A}^d$, where $e = \mathrm{ord}_\alpha(\mathrm{Jac}_{X^0})$ and $X^0 \subset X$ is the irreducible component containing α_η. Moreover, we get that the tangent cone of Z at z is the scheme-theoretic image of the map of tangent cones induced by $X_\infty \to Y_{2e-1}$.

Finally, in Section 3.6 we derive some consequences of the results of the previous sections to the study of the *Mather–Jacobian discrepancy*. We give a new proof of a theorem in [MR18] and combine our formula for the embedding codimension with a result in [BS19b] to show that $\mathbb{Q}$-factorial toric singularities are not *MJ-terminal*. This concludes the final chapter of this monograph.

We should mention here that large parts of this monograph were taken from the author's collaborative work together with H. Hauser in [CH17], with T. de Fernex and R. Docampo in [CdFD20], and finally with L. Narváez in [CN20]. We will indicate at the beginning of each chapter which sections were originally a part of these works.

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Chapter 1

Completion and rings of formal power series

The main purpose of this chapter is to extend basic techniques from commutative algebra over Noetherian rings to infinite-variate formal power series rings and their quotients. Our motivation for studying such rings is that they are the building blocks for formal completions of non-Noetherian schemes. As such, Section 1.1 is dedicated to an in-depth review of the definition of formal schemes which are not necessarily adic. In Sections 1.2 and 1.3 we introduce quasi-adic rings as rings endowed with a certain kind of filtration and prove an extension of the Cohen Structure theorem. Finally, Sections 1.4 to 1.6 are devoted to methods and results of classical commutative algebra adapted to infinite-variate formal power series rings. Parts of Section 1.2 as well as most of Sections 1.4 and 1.6 were taken from [CdFD20].

1.1 Affine formal schemes

If x is a point on a scheme X, then the ring-theoretic completion $\widehat{\mathcal{O}}_{X,x}$ of the local ring $\mathcal{O}_{X,x}$ describes the infinitesimal information of X at x. As is well-known, the corresponding colimit of thickenings $\operatorname{Spec}(\mathcal{O}_{X,x}/\mathfrak{m}_{X,x}^n)$ in the category of locally ringed spaces is no longer a scheme. Thus one should not consider the completion $\widehat{\mathcal{O}}_{X,x}$ as an element of the usual category of commutative rings. The key is to treat $\widehat{\mathcal{O}}_{X,x}$ as a *topological ring* with its limit topology; in particular, its only geometric point is x itself. The geometric object corresponding to $\widehat{\mathcal{O}}_{X,x}$ is called the *formal neighborhood* of x and is a particular case of a *formal scheme*. As the only formal schemes we will consider here in detail are such formal neighborhoods of singularities we can restrict our study to the topological rings determining them. The reader who is satisfied with this explanation may indeed skip this section entirely. However, the formal neighborhoods we are studying in Chapter 3 are non-Noetherian and non-adic. In this case, there are some intricacies not encountered when restricting to the Noetherian adic case, which we found worth exploring in detail. Our goal in this section is first to provide a definition of what we call *strongly admissible* formal schemes, which is functorial in nature and well-suited to our study of formal neighborhoods of arc spaces. Second, we will compare it to some of the other definitions in use throughout the literature. As we mostly consider only the affine case we will not mention how to construct the formal spectrum as a locally ringed space, as is done for example in [EGA I]. However, we will give a self-contained proof of the functorial properties we will make use of in Sections 3.4 and 3.5. This section should be considered purely expository, with various references mentioned throughout.

1.1.1 Inverse systems and inverse limits

Let us start by recalling the definition of an inverse limit in an arbitrary category. Let $(I, \leq)$ be a *directed set*, that is, a set I with a preorder $\leq$ such that each two elements have an upper bound. As possibly the most important example here, one should think of $I = \mathbb{N}$ with the usual total order. Every directed set I induces a small filtered category $\mathcal{I}$ which has as its objects just the elements of I and as morphisms $\mathrm{Hom}(i, j) = \{\star\}$ whenever $i \leq j$. The reader shall note that the results of this section generalize with only minor modifications to the case of a general small filtered category $\mathcal{I}$; the choice to restrict to directed sets was made only for convenience's sake.

Let $\mathcal{D}$ be any category. An *inverse system* $\underline{A}$ in $\mathcal{D}$ over I is a functor $\mathcal{I}^{\mathrm{op}} \to \mathcal{D}$. That is, an inverse system $\underline{A}$ consists of objects $A_i = \underline{A}(i)$ and morphisms $\pi_{i,j} : A_j \to A_i$ for $i \leq j$ such that $\pi_{i,k} = \pi_{i,j} \circ \pi_{j,k}$ for $i \leq j \leq k$ and $\pi_{i,i} = \mathrm{id}_{A_i}$. We call the $\pi_{i,j}$ *transition maps.*

An *inverse limit* is the categorical limit of an inverse system. More precisely, if $\underline{A}$ is an inverse system, then its limit is the universal cone over $\underline{A}$: there exist an object A and morphisms $\pi_i : A \to A_i$ such that $\pi_j = \pi_i \circ \pi_{i,j}$ for $i \leq j$ and for every other such (A', π'_i) there exists a unique morphism $A' \to A$. This can be expressed diagramatically as

$$\begin{tikzcd}
& A' \arrow[d, dashed] \arrow[ddl, "\pi'_j"'] \arrow[ddr, "\pi'_i"] & \\
& A \arrow[dl, "\pi_j"] \arrow[dr, "\pi_i"'] & \\
A_j \arrow[rr, "\pi_{i,j}"'] & & A_i.
\end{tikzcd}$$

We write $A = \lim_i A_i$. If $\lim \underline{A}$ exists in $\mathcal{D}$ for all inverse systems $\underline{A}$ then we say that $\mathcal{D}$ *has inverse limits.*

Example 1.1.1. The category of sets, which we denote by Set, has inverse limits. In fact, if A_i is any inverse limit of sets over some I, then its inverse limit $A = \lim_i A_i$ is given as

$$A = \{(a_i)_{i \in I} \in \prod_i A_i : \pi_{i,j}(a_j) = a_i \; \forall i \leq j\}.$$

Remark 1.1.2. The dual notion of an inverse limit is that of a *direct limit.* If $(I, \leq)$ is a directed set, then a *direct system* in a category $\mathcal{D}$ over I is a functor $\overline{A} : \mathcal{I} \to \mathcal{D}$. A direct limit is the categorical colimit of a direct system, that is, the universal cocone (A, ι_i) over some $\overline{A}$. This property can be again expressed conveniently by the commutative diagram

$$\begin{tikzcd}
A_i \arrow[rr, "\iota_{i,j}"] \arrow[dr, "\iota_i"] \arrow[ddr, "\iota_i'"'] & & A_j \arrow[dl, "\iota_j"'] \arrow[ddl, "\iota_j'"] . \\
& A \arrow[d, dashed] & \\
& A' &
\end{tikzcd}$$

The direct limit of a direct system $\overline{A}$ in the category Set of sets is given by

$$A = \coprod_i A_i / \sim,$$

where $a_i \sim a_j$ if there exists $k \geq i, j$ such that $\iota_{i,k}(a_i) = \iota_{j,k}(a_j)$.

Remark 1.1.3. An equivalent description of an inverse limit can be given as follows: consider the diagonal functor $\Delta : \mathcal{D} \to \mathcal{D}^{\mathcal{I}^{\mathrm{op}}}$ which associates to each $E \in \mathcal{D}$ the functor $\Delta(E) : \mathcal{I}^{\mathrm{op}} \to \mathcal{D}$ given by $\Delta(E)(i) = E$ and $\Delta(E)(j \to i) = \mathrm{id}_E$. For each inverse system $\underline{A}$ in $\mathcal{D}$ consider the functor

$$\lim_{\mathcal{D}} \underline{A} : \mathcal{D}^{\mathrm{op}} \to \mathrm{Set},\ E \mapsto \mathrm{Nat}(\Delta(E), \underline{A}).$$

Then the inverse limit of $\underline{A}$ exists if and only if $\lim_{\mathcal{D}} \underline{A}$ is representable, and we have

$$\mathrm{Hom}_{\mathcal{D}}(-, \lim_i A_i) \simeq \lim_{\mathcal{D}} \underline{A}.$$

Remark 1.1.4. Let $\underline{A}$ be an inverse system in $\mathcal{D}$ over a directed set I. For any $E \in \mathcal{D}$, composing $\underline{A}$ with $h_E = \mathrm{Hom}_{\mathcal{D}}(-, E)$ gives a direct system in Set, whose direct limit $\mathrm{colim}_i \mathrm{Hom}_{\mathcal{D}}(A_i, E)$ always exists. Now assume that the limit $A = \lim_i A_i$ of $\underline{A}$ exists in $\mathcal{D}$. Applying $\mathrm{Hom}_{\mathcal{D}}(-, E)$ to the corresponding universal cone in $\mathcal{D}$ yields a cocone in Set and thus a unique map

$$\operatorname*{colim}_i \mathrm{Hom}_{\mathcal{D}}(A_i, E) \to \mathrm{Hom}_{\mathcal{D}}(A, E). \tag{1.1a}$$

It is easily seen that this map is natural in E, thus we obtain a natural transformation

$$\operatorname*{colim}_i h^{A_i} \to h^A = \mathrm{Hom}_{\mathcal{D}}(A, -). \tag{1.1b}$$

In some cases, even though the category $\mathcal{D}$ itself has inverse limits, we are interested in inverse limits of $\mathcal{D}$ taken as the limit of the system embedded in some larger category $\mathcal{C}$. To make this precise, let $e : \mathcal{D} \to \mathcal{C}$ be a fully faithful functor between locally small[1] categories; we will often consider $\mathcal{D}$ as a full subcategory of $\mathcal{C}$, but it will be convenient at times to refer to e explicitly. Assume that $\mathcal{C}$ has inverse limits. Denote by $\underleftarrow{\mathcal{D}}_{\mathcal{C}}$ the full subcategory of $\mathcal{C}$ consisting of all objects isomorphic to $\lim_i A_i$ for some inverse system $\underline{A}$ in $\mathcal{D}$ over some directed set $(I, \leq)$. Our aim is to find conditions for when objects in $\underleftarrow{\mathcal{D}}_{\mathcal{C}}$ are determined by *their values in* $\mathcal{D}$. More precisely, consider the functor

$$\widehat{h} : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}^{\mathcal{D}},\ A \mapsto \widehat{h}^A := h^A \circ e,$$

[1] Locally small meaning that $\mathrm{Hom}_{\mathcal{C}}(A, B)$ is a set for all $A, B \in \mathcal{C}$.

where $h^A = \mathrm{Hom}_{\mathcal{C}}(A, -)$ as before. For each pair $A, B \in \mathcal{C}$ the functor $\widehat{h}$ gives a map

$$\widehat{h}_{A,B} : \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Nat}(\widehat{h}^B, \widehat{h}^A) \tag{1.1c}$$

which maps each morphism $A \to B$ to the restriction of the induced transformation $h^B \to h^A$ to $\mathcal{D}$. Clearly $\underleftarrow{\mathcal{D}}_{\mathcal{C}}$ is a subcategory of $\mathrm{Set}^{\mathcal{D}}$ via $\widehat{h}$ if and only if (1.1c) is a bijection for all $A, B \in \underleftarrow{\mathcal{D}}_{\mathcal{C}}$. For $A \in \underleftarrow{\mathcal{D}}_{\mathcal{C}}$ write $A = \lim_i A_i$ for some inverse system $\underline{A}$. By Remark 1.1.4 we have a natural transformation

$$\operatorname*{colim}_i h^{A_i} \to \widehat{h}^A. \tag{1.1d}$$

On the other hand a straightforward application of the Yoneda lemma yields the following.

Lemma 1.1.5. *Let $A, B \in \underleftarrow{\mathcal{D}}_{\mathcal{C}}$ and let $B = \lim_i B_i$ over some directed set $(I, \leq)$. Then:*

$$\mathrm{Hom}_{\mathcal{C}}(A, B) \simeq \mathrm{Nat}(\operatorname*{colim}_i h^{B_i}, \widehat{h}^A).$$

Proof. We have

$$\mathrm{Hom}_{\mathcal{C}}(A, \lim_i B_i) \simeq \lim_i \widehat{h}^A(B_i) \simeq \lim_i \mathrm{Nat}(h^{B_i}, \widehat{h}^A) \simeq \mathrm{Nat}(\operatorname{colim} h^{B_i}, \widehat{h}^A).$$

□

Corollary 1.1.6. *The map $\widehat{h}_{A,B}$ from* (1.1c) *is injective*[2] *for all $A, B \in \underleftarrow{\mathcal{D}}_{\mathcal{C}}$.*

Proof. Precomposition with the natural transformation (1.1d) gives a map

$$\mathrm{Nat}(\widehat{h}^B, \widehat{h}^A) \to \mathrm{Nat}(\operatorname{colim} h^{B_i}, \widehat{h}^A) \simeq \mathrm{Hom}_{\mathcal{C}}(A, B).$$

Note that the image of a natural transformation η under this map is $\lim_i \eta_{B_i}(\pi_i)$, with $\pi_i : B \to B_i$ being the natural projection. Thus in particular the image of $f : A \to B$ under the composition

$$\mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Nat}(\widehat{h}^B, \widehat{h}^A) \to \mathrm{Hom}_{\mathcal{C}}(A, B)$$

is $\lim_i \pi_i \circ f = f$, which implies that the above map is just the identity on $\mathrm{Hom}_{\mathcal{C}}(A, B)$. □

What is left to check is when the map $\widehat{h}_{A,B}$ is surjective. We will mention one particular case of interest here:

Proposition 1.1.7. *Let $\mathcal{D}, \mathcal{C}, \underleftarrow{\mathcal{D}}_{\mathcal{C}}$ and $\widehat{h}$ as before. Let $A \in \underleftarrow{\mathcal{D}}_{\mathcal{C}}$ and write $A = \lim_i A_i$. Then the map* (1.1a)

$$\operatorname*{colim}_i \mathrm{Hom}_{\mathcal{D}}(A_i, E) \simeq \mathrm{Hom}_{\mathcal{C}}(A, E)$$

is surjective for all $E \in \mathcal{D}$ if and only if every map $A \to E$ factors as $A \to A_i \to E$. If this property holds for all $A \in \underleftarrow{\mathcal{D}}_{\mathcal{C}}$ then the functor $\widehat{h} : \underleftarrow{\mathcal{D}}_{\mathcal{C}} \to \mathrm{Set}^{\mathcal{D}}$ is fully faithful.

[2] We should note here that in general $\mathrm{Nat}(\widehat{h}^B, \widehat{h}^A)$ will not be a set and thus injectivity should be understood as an 1-1 assignment of objects.

Proof. The first assertion is clear. For the second statement note that the assumption is equivalent to $\operatorname{colim}_i h^{A_i} \to \widehat{h}^A$ from (1.1d) being an epimorphism. Thus $\mathrm{Nat}(\widehat{h}^B, \widehat{h}^A) \to \mathrm{Nat}(\operatorname{colim} h^{B_i}, \widehat{h}^A)$ is injective, and since the identity on $\mathrm{Hom}_{\mathcal{C}}(A, B)$ factors as

$$\mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Nat}(\widehat{h}^B, \widehat{h}^A) \to \mathrm{Hom}_{\mathcal{C}}(A, B)$$

we get the claim. □

Note that the assumptions of Proposition 1.1.7 are quite restrictive on $\mathcal{D}$. However, as will be dicussed later, they are fulfilled if $\mathcal{D}$ is the full subcategory of discrete objects of $\mathcal{C}$ some category of algebraic objects (e.g. Ab or Ring) endowed with a topology that is sufficiently compatible with the algebraic structure.

Let us finish this section with a simple observation. A subcategory $\mathcal{D}$ of $\mathcal{C}$ is called a *discrete subcategory* of $\mathcal{C}$ if the embedding $e : \mathcal{D} \to \mathcal{C}$ has a faithful right adjoint $u : \mathcal{C} \to \mathcal{D}$. We then call u the *forgetful functor* associated to $\mathcal{D}$. Similar to above, the main example we are thinking about here is the inclusion of the category Set as the discrete objects into the category of topological spaces Top. The fact that an inverse limit of topological spaces is an inverse limit of underlying sets can be formulated as follows.

Lemma 1.1.8. *Let $\mathcal{D}$ be a discrete subcategory of $\mathcal{C}$ and $u : \mathcal{C} \to \mathcal{D}$ the forgetful functor right adjoint to the embedding $e : \mathcal{D} \to \mathcal{C}$. If $\mathcal{C}$ has inverse limits, then so has $\mathcal{D}$. Explicitly, for each inverse system $\underline{A}$ of $\mathcal{D}$ we have*

$$\lim_i A_i \simeq u(\lim_j e(A_i)).$$

Proof. Right adjoints commute with limits. □

1.1.2 Pro-objects

We will now present a slightly different approach to consider inverse limits "outside" a given category $\mathcal{D}$, that is, to consider the inverse system $\underline{A}$ itself as a (formal) limit. As a remark for the reader: throughout this section we will ignore any potential set-theoretic issues, which could all be remedied by a suitable choice of Grothendieck universe. Details can be found in [SGA 4_1, Expose I.8]. While the material of this section is technically not needed in its generality to give a definition of formal schemes, the conditions obtained in Proposition 1.1.15 are essentially those used to define affine formal schemes functorially in [MP74], and appear quite naturally here.

Some care has to be taken when choosing the correct notion of a morphism between inverse system. As the following example shows, considering only natural transformation between inverse systems $\underline{A}$ and $\underline{B}$ as functors $\mathcal{I}^{\mathrm{op}} \to \mathcal{D}$ is too restrictive.

Example 1.1.9. Let $I = (\mathbb{N}, \leq)$ and $\mathcal{D} = \mathrm{Ring}$ the category of commutative rings with 1. Define $\underline{A}$ to be the inverse system $A_i = \mathbb{Z}[t]/(t^{i+1})$, that is,

$$\underline{A} : \mathbb{Z} \leftarrow \mathbb{Z}[t]/(t^2) \leftarrow \mathbb{Z}[t]/(t^3) \leftarrow \dots$$

Now define another inverse system $\underline{B}$ via $B_i = A_{i-1}$, $B_0 = k$ and $\pi^B_{0,1} = \mathrm{id}_{\mathbb{Z}}$. Written as above:

$$\underline{B} : \mathbb{Z} \leftarrow \mathbb{Z} \leftarrow \mathbb{Z}[t]/(t^2) \leftarrow \dots$$

Clearly $\lim A_i \simeq \lim B_j \simeq \mathbb{Z}[[t]]$, but $\underline{A}$ and $\underline{B}$ are not isomorphic when considered as functors $\mathcal{I}^{\mathrm{op}} \to \mathcal{D}$. To see this, note that the only morphism $A_1 = \mathbb{Z}[t]/(t^2) \to \mathbb{Z} = B_1$ given by $t \mapsto 0$ does not lift to an isomorphism of the limits.

As in the last section, every inverse system $\underline{A}$ gives rise to a functor $\mathrm{colim}_i\, h^{A_i}$ from $\mathcal{D}$ to Set. Let $\underline{A}$ and $\underline{B}$ be inverse systems in $\mathcal{D}$ over directed sets I and J. By the Yoneda lemma, we have

$$\mathrm{Nat}(\operatorname*{colim}_j h^{B_j}, \operatorname*{colim}_i h^{A_i}) \simeq \lim_j \mathrm{Nat}(h^{B_j}, \operatorname*{colim}_i h^{A_i}) \simeq \lim_j \operatorname*{colim}_i \mathrm{Hom}_{\mathcal{D}}(A_i, B_j). \tag{1.1e}$$

This leads us to make the following definition:

Definition 1.1.10. A *pro-object* of $\mathcal{D}$ is an inverse system $\underline{A}$ in $\mathcal{D}$ over some (variable) directed set $\mathcal{I}$. The category of pro-objects of $\mathcal{D}$, denoted by $\mathrm{Pro}(\mathcal{D})$, has its morphisms defined by the rule

$$\mathrm{Hom}_{\mathrm{Pro}(\mathcal{D})}(\underline{A}, \underline{B}) = \lim_j \operatorname*{colim}_i \mathrm{Hom}_{\mathcal{D}}(A_i, B_j).$$

Composition of two morphisms is defined as composition in $\mathrm{Set}^{\mathcal{D}}$ via the identification (1.1e).

Let us elucidate the last part a bit using the explicit descriptions of limits and colimits of sets presented in Example 1.1.1 and Remark 1.1.2. A morphism of two inverse systems $\underline{A}$ and $\underline{B}$ over I, J is represented by a family $f = (f_j)_{j\in J}$, where $f_j : A_{i_j} \to B_j$ for some $i_j \in I$, satisfying $\pi^B_{j,j'} \circ f_{j'} \circ \pi^A_{i_{j'},k} = f_j \circ \pi^A_{i_j,k}$ for $j \le j'$ and some $k \ge i_j, i_{j'}$. Two such families f, f' are equivalent if, for each $j \in J$, there exists $k_j \ge i_j, i'_j$ such that $f_j \circ \pi^A_{i_j,k_j} = f'_j \circ \pi^A_{i'_j,k_j}$. To compose two morphisms $\underline{A} \to \underline{B}$ and $\underline{B} \to \underline{C}$ we choose representations $f = (f_j)_{j\in J}$ and $g = (g_k)_{k\in K}$. For each $k \in K$ we set

$$(g \circ f)_k := g_k \circ f_{j_k} : A_{i_{j_k}} \to C_k.$$

As a way of illustrating composition of morphisms in $\mathrm{Pro}\,\mathcal{D}$ the reader might find the following mnemonic diagram for the case $I, J = \mathbb{N}$ helpful:

$$\begin{tikzcd}
A_0 & \dots \arrow[l] & A_{j_0} \arrow[l] & \dots \arrow[l] & A_{i_{j_0}} \arrow[l] \arrow[dll, "f_{j_0}"'] & \dots \arrow[l] \\
B_0 & \dots \arrow[l] & B_{j_0} \arrow[l] \arrow[dll, "g_0"'] & \dots \arrow[l] & B_{i_{j_0}} \arrow[l] & \dots \arrow[l] \\
C_0 & \dots \arrow[l] & C_{j_0} \arrow[l] & \dots \arrow[l] & C_{i_{j_0}} \arrow[l] & \dots \arrow[l]
\end{tikzcd}$$

Let $\varphi : I' \to I$ be an order-preserving map between directed sets. Given any pro-object $\underline{A}$ over I we obtain a pro-object $\underline{A}'$ over I' by precomposing with $\mathcal{I}' \to \mathcal{I}$. We call $\underline{A}'$ the pro-object obtained by *change of index*. Note that

φ canonically induces a morphism $\underline{A}' \to \underline{A}$. If φ is *cofinal*, that is, for every $i \in I$ there exists $i' \in I'$ with $i \leq \varphi(i')$, then it is easy to see that $\underline{A}' \to \underline{A}$ is an isomorphism in $\mathrm{Pro}(\mathcal{D})$. In particular, given any morphism $f : \underline{A} \to \underline{B}$ we may find a change of index $\underline{A}' \to \underline{A}$ such that the composition $f' : \underline{A}' \to \underline{A}$ is represented by a family

$$f'_d : A_d \to B_d.$$

By construction the assignment $\underline{A} \to \mathrm{colim}_i\, h^{A_i}$ gives a fully faithful functor $L : \mathrm{Pro}(\mathcal{D})^{\mathrm{op}} \to \mathrm{Set}^{\mathcal{D}}$. Any functor $\mathcal{D} \to \mathrm{Set}$ which is isomorphic to $L(\underline{A})$ for some $\underline{A} \in \mathrm{Pro}(\mathcal{D})$ is called *pro-representable*.

Every $E \in \mathcal{D}$ can clearly be considered as a pro-object over the directed set $I = \{\star\}$. This gives a fully faithful functor $c : \mathcal{D} \to \mathrm{Pro}(\mathcal{D})$, hence we may treat $\mathcal{D}$ as a subcategory of $\mathrm{Pro}(\mathcal{D})$. Note that the composition $L \circ c$ above is not equal to $E \to h^E = \mathrm{Hom}_{\mathcal{D}}(E, -)$, but only isomorphic to it.

By construction, for every inverse system $\underline{A}$ in $\mathcal{D}$, the inverse limit of $\underline{A}$ exists in $\mathrm{Pro}(\mathcal{D})$ and is isomorphic to $\underline{A}$ as a pro-object. Unfortunately, in general this limit will be different from $\lim_i A_i$ in $\mathcal{D}$. The following example is attributed to P. Gabriel in [MP74].

Example 1.1.11. Let I be the set of cofinite sets of prime numbers and, for $S, S' \in I$ declare $S \leq S'$ whenever $S' \subset S$. For each such $S \in I$ we consider the ring $\mathbb{Q}_S$ of rational numbers whose denominator has prime factors only in S (i.e. $\mathbb{Q}_S$ is the localization of $\mathbb{Z}$ with respect to the multiplicative closure of S). Then $G = (\mathbb{Q}_S)$ is an inverse system of the category Ring. For every $R \in \mathrm{Ring}$

$$h^{\mathbb{Q}_S}(R) = \mathrm{Hom}_{\mathrm{Ring}}(\mathbb{Q}_S, R) = \begin{cases} \star, & \text{every } p \in S \text{ is a unit in } R \\ \emptyset, & \text{otherwise,} \end{cases}$$

thus in particular

$$L(G)(\mathbb{Z}) = \operatorname*{colim}_S \mathrm{Hom}_{\mathrm{Ring}}(\mathbb{Q}_S, \mathbb{Z}) = \emptyset.$$

But clearly $\lim_S \mathbb{Q}_S = \mathbb{Z}$ in Ring. Note that the projection maps $\mathbb{Z} \to \mathbb{Q}_S$ are epimorphisms and I is countable.

Similar to our main result in the last section, we want to find conditions on when pro-objects of some category $\mathcal{D}$ agree with the limit of the underlying system in some larger category $\mathcal{C}$. To make this precise, note that for each inverse system $\underline{A}$ we have an isomorphism of functors from $\mathcal{D}$ to Set

$$\mathrm{Hom}_{\mathrm{Pro}(\mathcal{D})}(c(-), \underline{A}) \simeq \mathrm{Nat}(\Delta(-), \underline{A}),$$

where Δ is defined as in Remark 1.1.3. Thus the limit of $\underline{A}$ exists in $\mathcal{D}$ if and only if $\mathrm{Hom}_{\mathrm{Pro}(\mathcal{D})}(c(-), \underline{A})$ is representable. This in turn implies that there exists a partial right adjoint to $c : \mathcal{D} \to \mathrm{Pro}(\mathcal{D})$, defined on the full subcategory consisting of those $\underline{A}$ such that the above functor is representable, and mapping $\underline{A}$ to its limit $\lim_i A_i$. If $\mathcal{C}$ now is a category which has inverse limits, by abuse of notation we denote the (globally defined) right adjoint to c as

$$\lim_{\mathcal{C}} : \mathrm{Pro}(\mathcal{C}) \to \mathcal{C},\ \underline{A} \mapsto \lim_i A_i.$$

Note that $\lim_{\mathcal{C}} \circ c \simeq \mathrm{id}_{\mathcal{C}}$ (but not equality!). Any functor $f : \mathcal{D} \to \mathcal{C}$ clearly induces a functor $\mathrm{Pro}(f) : \mathrm{Pro}(\mathcal{D}) \to \mathrm{Pro}(\mathcal{C})$, which is fully faithful if and only if f is. Given a fully faithful $e : \mathcal{D} \to \mathcal{C}$, consider the composition

$$\underline{e} := \lim_{\mathcal{C}} \circ \mathrm{Pro}(e) : \mathrm{Pro}(\mathcal{D}) \to \mathcal{C},\ \underline{A} \mapsto \lim_i A_i.$$

Lemma 1.1.12 ([SGA 4_1, Proposition 8.7.5.a)]). *The functor $\underline{e}$ is fully faithful if and only if the natural map* (1.1a)

$$\operatorname*{colim}_i \operatorname{Hom}_{\mathcal{D}}(A_i, E) \to \operatorname{Hom}_{\mathcal{C}}(\lim_i A_i, E)$$

is a bijection for all $\underline{A} \in \operatorname{Pro}(\mathcal{D})$ and $E \in \mathcal{D}$.

Proof. This is a tautology. □

We already established surjectivity of (1.1a) for a special case in Proposition 1.1.7, so let us now determine when this map is injective. We start with an easy observation.

Lemma 1.1.13. *Let $\underline{A} \in \operatorname{Pro}(\mathcal{D})$ and assume its limit $A = \lim_i A_i$ exists in $\mathcal{C}$.*

1. *If the projections $\pi_i : A \to A_i$ are epimorphisms, then*

$$\operatorname*{colim}_i \operatorname{Hom}_{\mathcal{D}}(A_i, E) \to \operatorname{Hom}_{\mathcal{C}}(A, E) \tag{1.1f}$$

   is injective for all $E \in \mathcal{D}$.

2. *Assume all transition maps $\pi_{i,j} : A_j \to A_i$ are epimorphisms. If the map* (1.1f) *is injective for all $E \in \mathcal{D}$, then $\pi : A \to A_i$ is an epimorphism for all $i \in I$.*

Proof. To prove (1), let $g_1 : A_i \to E$ and $g_2 : A_j \to E$ be two morphisms such that $g_1 \circ \pi_i = g_2 \circ \pi_j$. Let k be such that $k \geq i, j$, then

$$g_1 \circ \pi_{i,k} \circ \pi_k = g_2 \circ \pi_{j,k} \circ \pi_k.$$

Since π_k is an epi, we get that g_1 and g_2 are equivalent in $\operatorname{colim}_i \operatorname{Hom}_{\mathcal{D}}(A_i, E)$.

For assertion (2), let $g_1, g_2 : A_i \to B$ be two morphisms such that $g_1 \circ \pi_i = g_2 \circ \pi_i$. By assumption there exists $k \geq i$ with $g_1 \circ \pi_{i,k} = g_2 \circ \pi_{i,k}$ and since $\pi_{i,k}$ is an epi we have $g_1 = g_2$.

□

Observe that the projection maps π_i being epimorphisms implies that the transition maps are too. We call an inverse system $\underline{A}$ such that all transition maps $\pi_{i,j}$ are epimorphisms a *strict pro-object*. If $\mathcal{D}$ is a concrete category, then $\underline{A}$ is called a *surjective system* if the map on underlying sets induced by $\pi_{i,j}$ is surjective. In the case where I is countable, the following result is well-known:

Lemma 1.1.14. *If I is a countable directed set and $\underline{A}$ an inverse system of* Set*; write $A = \lim_i A_i$ for its limit. If all transition maps $\pi_{i,j}$ are surjective, then so are the projection maps $\pi_i : A \to A_i$.*

Proof. For a countable set I and $i \in I$ we can always construct a cofinal subsequence starting with i. Applying the corresponding change of index we may assume $I = \mathbb{N}$. But then the assertion is clear by the construction in Example 1.1.1. □

This immediately gives the same result for many "algebraic" categories like Ab and Ring whose inverse limit is constructed by endowing the inverse limit of underlying sets with the respective operations. Unfortunately, as shown e.g. in [Wat72], the inverse limit of a surjective system of nonempty sets over an uncountable directed set I might be empty! Thus, even when substituting the stronger notion of surjective system, we cannot extend Lemma 1.1.13 to a more general characterization.

Let us summarize the main result of this section thus as follows:

Proposition 1.1.15. *Let $\mathcal{D} \to \mathcal{C}$ be a fully faithful functor and assume $\mathcal{C}$ has inverse limits. For each $\underline{A} \in \mathrm{Pro}(\mathcal{D})$ write $A := \lim_i A_i$ for its limit in $\mathcal{C}$. Let $\mathcal{P} \subset \mathrm{Pro}(\mathcal{D})$ be a full subcategory satisfying*

(A) for each $\underline{A} \in \mathcal{P}$ the projection maps $\pi_i : A \to A_i$ are epimorphisms in $\mathcal{C}$.

(B) for each $\underline{A} \in \mathcal{P}$ every map $A \to E$ with $E \in \mathcal{D}$ factors as $A \to A_i \to E$ for some i.

Then we have that

$$\operatorname{colim} h^{A_i} \simeq \widehat{h}^A$$

as functors $\mathcal{D} \to \mathrm{Set}$ and this induces an equivalence of $\mathcal{P}$ with the full subcategory of all $B \in \mathcal{C}$ with $B \simeq \lim_i A_i$ for some $\underline{A} \in \mathcal{P}$.

Proof. Combine Proposition 1.1.7 with Lemma 1.1.13. □

To finish, consider the case of abelian categories, where we may refine the second assertion of Lemma 1.1.13 slightly. Let us first recall the definition of a Mittag-Leffler system:

Definition 1.1.16. Let $\mathcal{A}$ be an abelian category an $\underline{A}$ an inverse system in $\mathcal{A}$. Then $\underline{A}$ is called *Mittag-Leffler* if, for each $i \in I$ there exists a $k \geq i$ such that, for all $j \geq k$, we have:

$$\mathrm{Im}(\pi_{i,j} : A_j \to A_i) \simeq \mathrm{Im}(\pi_{i,k} : A_k \to A_i).$$

The Mittag-Leffler property is often stated as the sequence $(\mathrm{Im}(\pi_{i,j}))_{j \geq i}$ stabilizing for each $i \in I$.

Proposition 1.1.17. *Assume $\mathcal{C}$ has inverse limits and let $\mathcal{A}$ be an abelian discrete subcategory of $\mathcal{C}$. If $\underline{A} \in \mathrm{Pro}(\mathcal{A})$ such that*

$$\operatorname*{colim}_i \mathrm{Hom}_{\mathcal{A}}(A_i, E) \to \mathrm{Hom}_{\mathcal{C}}(\lim_i A_i, E)$$

is injective for all $E \in \mathcal{A}$, then $\underline{A}$ is Mittag-Leffler.

Proof. By assumption (see Lemma 1.1.8) $\mathcal{A}$ has inverse limits and after proper identification we have

$$\mathrm{Hom}_{\mathcal{C}}(\lim_i A_i, E) \simeq \mathrm{Hom}_{\mathcal{A}}(\lim_i A_i, E).$$

Fix $i \in I$ and set $I_i := \{j \in I : i \leq j\}$. Then $B_{ij} := \mathrm{Im}(\pi_{i,j})$ forms an inverse system over I_i. Consider its limit B_i in $\mathcal{A}$. Then $A \to A_i$ factors as $A \to B_i \to A_i$ in $\mathcal{A}$. Letting $C_i := \mathrm{coker}(B_i \to A_i)$ we thus have that the composition $A \to C_i$ is 0. Consider now the canonical map $A_i \to C_i$; since its image in $\mathrm{Hom}_{\mathcal{C}}(\lim_i A_i, C_i)$ is the same as $0 : A_i \to C_i$, by assumption there exists $k \geq i$ such that $\pi_{i,k}$ factors as $A_k \to B_i \to A_i$. This implies, for $j \geq k$, that $\mathrm{Im}(\pi_{i,j}) = \mathrm{Im}(\pi_{i,k})$. □

For many abelian categories such as Ab the limit of any countable Mittag-Leffler system will have its projection maps being epimorphisms. However, this is not true in general - for a particularly infamous example consider [Nee02] in which a theorem on the exactness of the inverse limit functor proven 40 years earlier in [Roo61] was found to be erroneous.

1.1.3 Inverse limits of rings and completion

Let us start by recalling some definitions as per [EGA I]. A ring A is called a *topological ring* if it is endowed with a topology such that both operations $A \times A \to A$ are continuous. Homomorphism between topological rings are just continuous ring homomorphisms. A topological ring A is called *linearly topologized* if there exists a fundamental system of neighborhoods of 0 consisting of ideals. In that case, an ideal $\mathfrak{a}$ of A is called an *ideal of definition* if $\mathfrak{a}$ is open and every neighborhood of 0 contains some power $\mathfrak{a}^n$. If such an ideal of A exists then we say that A is *preadmissible.*

If k and A are topological rings, then A is a *topological k-algebra* if there exists a continous $k \to A$. A homomorphism between topological k-algebras is a continous k-algebra homomorphism.

Definition 1.1.18. We denote the category of topological rings by TRing and the full subcategory of linearly topologized rings by LTRing.

Observe that the fully faithful embedding $e : \mathrm{Ring} \to \mathrm{TRing}$ which endows each ring with its discrete topology is left adjoint to the forgetful functor $u : \mathrm{TRing} \to \mathrm{Ring}$.

Proposition 1.1.19. *1. The categories* Ring *and* TRing *have inverse limits. In both cases, the limit of a system $\underline{A}$ is given by endowing $\lim_i u(A_i)$ in* Set *with componentwise operations resp. the limit topology.*

2. Let $\underline{A}$ be an inverse system of discrete rings. Then $\lim A_i$ is a linearly topologized ring whose topology is separated.

3. Assume $\underline{A}$ is an surjective system over I countable. Then the projections $\pi_j : \lim_i A_i \to A_j$ are surjective.

Proof. (1) is clear. To see (2) note that a fundamental system of neighborhoods for 0 is given by $\ker(\pi_i)$. The third statement follows from Lemma 1.1.14. □

Let $A \in \mathrm{LTRing}$ and choose a neighborhood basis $\mathcal{U}$ for 0 consisting of open ideals. Note that $\mathcal{U}$ is a directed set with respect to inclusion, that is, $U \leq V$ if $V \subset U$. We define the *completion* of A to be the inverse limit

$$\widehat{A} := \lim_{U \in \mathcal{U}} A/U.$$

This definition is independent of the choice of $\mathcal{U}$, as any two neighborhood bases for 0 are cofinal for neighborhood filter for 0. Since each A/U is a discrete topological ring, we obtain that $\widehat{A}$ is linearly topologized and separated. We call A *complete* if $A \simeq \widehat{A}$. Clearly any $\widehat{A}$ is itself complete. Moreover, $\widehat{A}$ is the Cauchy completion of A and any A is complete if and only if it is Cauchy-complete. The category of complete topological rings will be denoted by CTRing. Concluding the suite of definitions from [EGA I] we call a preadmissible ring A *admissible* if A is complete.

Note that the natural homomorphism $A \to \widehat{A}$ is injective if and only if A is separated. Moreover, for any discrete E we have that

$$\mathrm{Hom}_{\mathrm{LTRing}}(A, E) \to \mathrm{Hom}_{\mathrm{LTRing}}(\widehat{A}, E)$$

is a bijection.

Let $A \to B$ and $A \to C$ be morphisms of topological rings. We endow the tensor product $B \otimes_A C$ of underlying rings with the final topology with respect to the canonical maps $B \to B \otimes_A C$, $C \to B \otimes_A C$. If A, B and C are in addition linearly topologized, then a fundamental system of neighborhoods for $0 \in B \otimes_A C$ is given by

$$\mathrm{Im}(\mathfrak{b} \otimes_A C + B \otimes_A \mathfrak{c}) \subset B \otimes_A C,$$

with $\mathfrak{b}$ and $\mathfrak{c}$ being open ideals in B resp C. In particular $B \otimes_A C \in \mathrm{LTRing}$.

Definition 1.1.20. We write $B \widehat{\otimes}_A C$ for the completion of $B \otimes_A C$ and call it the *completed tensor product* of B and C over A.

Proposition 1.1.21 ([EGA I, 0, (7.7.6)])**.** *Let $A \to B$, $A \to C$ be morphisms in* CTRing*. Then $B \widehat{\otimes}_A C$ is the pushout in the category* CTRing*, that is, the square*

$$\begin{array}{ccc} B \widehat{\otimes}_A C & \longleftarrow & B \\ \uparrow & & \uparrow \\ C & \longleftarrow & A \end{array}$$

is cocartesian. In particular, if CTRing_k *denotes the category of complete topological k-algebras, then the coproduct in* CTRing_k *is given by* $\widehat{\otimes}_k$*.*

Let us now apply the results of the last section to see that every complete topological ring is determined by maps into *discrete* topological rings:

Proposition 1.1.22. *The functor* $\mathrm{CTRing} \to \mathrm{Set}^{\mathrm{Ring}}$ *given by*

$$A \mapsto \widehat{h}^A = \mathrm{Hom}_{\mathrm{CTRing}}(A, e(-))$$

is fully faithful. Moreover, writing $A = \lim_{U \in \mathcal{U}} A/U$ *and* $\underline{A} := (A/U)_U \in \mathrm{Pro}(\mathrm{Ring})$ *we get*

$$\mathrm{Hom}_{\mathrm{CTRing}}(A, e(-)) \simeq \operatorname*{colim}_U \mathrm{Hom}_{\mathrm{Ring}}(A_U, -).$$

Proof. For any $f : A \to E$ with E discrete we have that $\ker(f)$ is open, thus contains an element $U \in \mathcal{U}$ and therefore f factors as $A \to A/U \to E$. Thus we can apply Proposition 1.1.7 and Proposition 1.1.15 and we are done. □

Elements in Pro(Ring) are often called *prorings*. As we have seen the category Pro(Ring) is not equivalent to CTRing. Write $\mathrm{Surj}_{\aleph_0}(\mathrm{Ring})$ for the full subcategory of surjective systems over a *countable* directed set I and $\mathrm{CTRing}_{\aleph_0}$ for the category of complete first-countable topological rings.

Lemma 1.1.23. *The functor* $\mathrm{Surj}_{\aleph_0}(\mathrm{Ring}) \to \mathrm{CTRing}_{\aleph_0}$ *given by* $\underline{A} \mapsto \lim_i A_i$ *is an equivalence of categories.*

Proof. The limit of each $\underline{A} \in \mathrm{Surj}_{\aleph_0}(\mathrm{Ring})$ has surjective projectio maps by Lemma 1.1.14 and thus we can apply Proposition 1.1.15 for $\mathcal{P} = \mathrm{Surj}_{\aleph_0}(\mathrm{Ring})$. □

1.1.4 Formal schemes

Motivated by the discussions in the previous section we can give the following minimal definition:

Definition 1.1.24. Let CTRing be as before the category of complete topological rings. For each $A \in \mathrm{CTRing}$ write $A = \lim_U A/U$. Then its *formal spectrum* $\mathrm{Spf}(A)$ is defined to be the functor

$$\widehat{h}^A = \mathrm{Hom}_{\mathrm{CTRing}}(A, e(-)) \simeq \operatorname*{colim}_U \mathrm{Hom}_{\mathrm{Ring}}(A/U, -) : \mathrm{Ring} \to \mathrm{Set},$$

where $e : \mathrm{Ring} \to \mathrm{CTRing}$ denotes the embedding of rings as a discrete topological rings in CTRing. A *affine formal scheme* is a functor $\mathcal{X} : \mathrm{Ring} \to \mathrm{Set}$ which is isomorphic to $\mathrm{Spf}(A)$ for some $A \in \mathrm{CTRing}$.

By construction, the assignment $A \mapsto \mathrm{Spf}(A)$ defines an equivalence between $\mathrm{CTRing}^{\mathrm{op}}$ and the category of affine formal schemes. Note that the category of affine schemes is equivalent to the category of representable functors $\mathrm{Ring} \to \mathrm{Set}$. Thus any affine formal scheme $\mathrm{Spf}(A)$ is a direct limit of affine schemes $\mathrm{Spec}(A/U)$.

Corollary 1.1.25. *The fiber product in the category of affine formal schemes exists and is given by* $\mathrm{Spf}(B) \times_{\mathrm{Spf}(A)} \mathrm{Spf}(C) = \mathrm{Spf}(B \widehat{\otimes}_A C)$.

The construction of the formal spectrum extends to a functor $\mathrm{LTRing}^{\mathrm{op}} \to \mathrm{Set}^{\mathrm{Ring}}$ via $A \mapsto \widehat{h}^A$. We note that this functor has a left adjoint which we shall quickly describe. Denote by $\mathbb{A}^1$ the forgetful functor $\mathrm{Ring} \to \mathrm{Set}$ and for each $\mathcal{X} : \mathrm{Ring} \to \mathrm{Set}$ the set[3] of natural transformations $\mathcal{O}_{\mathcal{X}} := \mathrm{Nat}(\mathcal{X}, \mathbb{A}^1)$. Note that $\mathcal{O}_{\mathcal{X}}$ is a ring under componentwise addition and multiplication, that is, for $\eta, \epsilon \in \mathrm{Nat}(\mathcal{X}, \mathbb{A}^1)$ we define

$$(\eta + \epsilon)_A := (x \mapsto \eta_A(x) + \epsilon_A(x)) : \mathcal{X}(A) \to A$$

and so on. For every $A \in \mathrm{Ring}$ and $f \in \mathcal{X}(A)$ we define a homomorphism $\tilde{f} : \mathcal{O}_{\mathcal{X}} \to A$ via $\tilde{f}(\eta) := \eta_A(f)$. As the category of affine schemes is filtered we see that the set of all ideals $\ker(\tilde{f})$ forms a direct system. Thus we can define a unqiue linear topology on $\mathcal{O}_{\mathcal{X}}$ with a fundamental system of neighborhoods of 0 given by $\ker(\tilde{f})$.

Proposition 1.1.26 ([MP74, Proposition 1.1])**.** *The functors $A \to \widehat{h}^A$ and $\mathcal{X} \mapsto \mathcal{O}_{\mathcal{X}}$ are adjoint on the right, that is,*

$$\mathrm{Nat}(\mathcal{X}, \widehat{h}^A) \simeq \mathrm{Hom}_{\mathrm{LTRing}}(A, \mathcal{O}_{\mathcal{X}}).$$

The counit of this adjunction is just the completion map $A \mapsto \widehat{A}$. Moreover, $\mathrm{Spf}(A) \simeq \mathrm{Spf}(\widehat{A})$ *for any* $A \in \mathrm{LTRing}$.

Remark 1.1.27. This is the approach of [Str99] (where affine formal schemes are called *solid*) and [MP74]. In the latter an affine formal scheme is defined as a functor satisfying conditions which are essentially equivalent to those of Proposition 1.1.15, thus making the restriction of the adjunction in Proposition 1.1.26 into an equivalence of categories.

[3] A suitable choice of Grothendieck universe is needed here.

Remark 1.1.28. A slightly different approach can be found in [BD]. There an *ind-scheme* is defined to be a direct limit of quasi-compact schemes X_i such that the transition maps $X_i \mapsto X_j$ are closed immersions. A formal scheme is an ind-scheme $\mathcal{X}$ such that $\mathcal{X}_{\mathrm{red}} = \mathrm{colim}_i(X_i)_{\mathrm{red}}$ is a scheme or equivalently, if the kernel of $\mathcal{O}_{X_j} \to \mathcal{O}_{X_i}$ consists only of nilpotent elements. A formal scheme $\mathcal{X}$ is affine if $\mathcal{X}_{\mathrm{red}}$ is, or equivalently, if every X_i is affine. Thus in particular the transition maps $\mathcal{O}_{X_j} \to \mathcal{O}_{X_i}$ are required to be surjective. If the directed set I over which the direct limit is taken is uncountable, then we have already seen that we cannot identify the inverse system $\mathcal{O}_{X_i}$ with its limit in TRing in general. In fact, in [BD, p. 7.12.17] there is a hint on how to construct a counterexample using (non-strict) Mittag-Leffler modules.

Remark 1.1.29. As a word of warning we should mention that an affine formal scheme in our very general sense is not necessarily an affine formal scheme in the sense of [EGA I], as the existence of an ideal of definition is not required. In particular, the construction of Spf as a locally ringed space may not be carried out in the same way as in [EGA I, I, (10.1.1)].

A less general but for our purposes entirely sufficient definition of formal schemes may be given as follows:

Definition 1.1.30. An $\aleph_0$-*formal scheme* $\mathcal{X}$ is a countable direct limit of schemes X_i such that all transition maps $X_j \to X_i$ are closed immersions. If in addition $\ker(\mathcal{O}_{X_i} \to \mathcal{O}_{X_j})$ is nilpotent for all $i \leq j$ we call $\mathcal{X}$ *strongly admissible*. In both cases $\mathcal{X}$ is *affine* if each X_i is affine.

Note that if $\mathcal{X}$ is an affine $\aleph_0$-formal scheme then $\mathcal{X} \simeq \mathrm{Spf}(\lim_i \mathcal{O}_{X_i})$, hence this definition is compatible with Definition 1.1.24. In fact:

Corollary 1.1.31. *The adjoint pair* $(\mathrm{Spf}, \mathcal{O})$ *induces an anti-equivalence between the category of affine* $\aleph_0$*-formal schemes and* $\mathrm{CTRing}_{\aleph_0}$*. Moreover, for* $A \in \mathrm{CTRing}_{\aleph_0}$*, we have that* $\mathrm{Spf}(A)$ *is strongly admissible if and only of* A *is admissible.*

Proof. The first part follows from Lemma 1.1.23. For the second assertion write $A = \lim_{i \in I} A_i$. As the directed set I is countable we may find a cofinal subsequence and thus assume $I = \mathbb{N}$. Then it is easy to see that $\ker(A \to A_0)$, where $A_0 = \mathcal{O}_{X_0}$, is an ideal of definition (see also [EGA I, pp. 0, 7.2.2]). □

Definition 1.1.32. An admissible $A \in \mathrm{CTRing}_{\aleph_0}$ will be called *strongly admissible*. Similarly, a preadmissible $A \in \mathrm{LTRing}_{\aleph_0}$ is called *strongly preadmissible.*

Corollary 1.1.33. *If* $B \to A$*,* $C \to A$ *are morphisms of strongly admissible rings, then* $B \widehat{\otimes}_A C$ *is again strongly admissible. In particular, the category of strongly admissible affine formal schemes has fiber products.*

Proof. If B and C are first countable, then so is $B \widehat{\otimes}_A C$. □

Every formal neighborhood in the following sense is a strongly admissible formal scheme:

Definition 1.1.34. If X is an affine scheme and $Z \subset X$ a closed subscheme defined by an ideal $\mathfrak{a} \subset \mathcal{O}_X$, then the *formal completion* $\widehat{X}_Z$ of X at Z is defined as $\mathrm{Spf}(\widehat{\mathcal{O}}_X)$, where $\widehat{\mathcal{O}}_X = \lim_i \mathcal{O}_X/\mathfrak{a}^i$. If $Z = \{x\}$ is a closed point of X, then $\widehat{X}_x$ is called the *formal neighborhood* of X at x.

Corollary 1.1.35. *If $\mathcal{O}_X$ is considered as a topological ring endowed with the $\mathfrak{a}$-adic topology, then*

$$\widehat{X}_Z \simeq \widehat{h}^{\mathcal{O}_X} \simeq \operatorname*{colim}_i \operatorname{Hom}_{\mathrm{Ring}}(\mathcal{O}_x/\mathfrak{a}^i, -).$$

To conclude this section, let us consider the case of formal neighborhoods of k-rational points. Let k be a field and write Nil_k for the category of local k-algebras $(R, \mathfrak{m})$ with $\mathfrak{m}$ nilpotent and such that $k \to R/\mathfrak{m}$ is an isomorphism. Elements of Nil_k will sometimes be called *test rings.* Note that any homomorphism between test rings is local and thus continuous for the $\mathfrak{m}$-adic topology.

Corollary 1.1.36. *The formal neighborhood $\widehat{X}_x$ of a k-scheme X at a k-rational point $x \in X(k)$ is determined by the functor*

$$\operatorname{Hom}_k(\mathcal{O}_{X,x}, -) : \mathrm{Nil}_k \to \mathrm{Set},$$

that is, we have $\widehat{X}_x \simeq \widehat{Y}_y$ if and only if there exists a isomorphism of the corresponding functors.

Proof. Clearly $\widehat{\mathcal{O}}_{X,x}$ is an inverse limit of test-rings, thus the statement follows either from the preceding discussion or directly by Proposition 1.1.7. □

Remark 1.1.37. For any test ring $(R, \mathfrak{m})$ a map $\tilde{x} : \mathcal{O}_{X,x} \to R$ is equivalent to a diagram

$$\begin{tikzcd} \mathrm{Spec}(R) \arrow[r, "\tilde{x}"] & X \\ \mathrm{Spec}(k) \arrow[u] \arrow[ur, "x"'] & \end{tikzcd},$$

where the vertical morphism is given by the quotient map $R \to k$. The morphism $\tilde{x}$ should be thought of as an infinitesimal deformation of x.

1.2 Formal power series rings and filtrations

The most commonly considered topological rings in commutative algebra are *adic topological rings*: if A is a ring and $\mathfrak{a}$ an ideal of A, then the $\mathfrak{a}$-*preadic* topology on A is defined by declaring the powers $\mathfrak{a}^n$, $n \in \mathbb{N}$ as a fundamental system of neighborhoods for $0 \in A$. If the topology on A is in addition separated and complete then A is called *adic*. In the terminology of last section, every preadic ring is obviously preadmissible and every adic ring admissible.

For Noetherian rings the (pre)adic topology is well-behaved. For example, for any Noetherian *local* $(A, \mathfrak{m})$ with the $\mathfrak{m}$-preadic topology its completion $\widehat{A} = \lim_i A/\mathfrak{m}^i$ is again Noetherian local with maximal ideal $\mathfrak{m}\widehat{A}$. Moreover, the limit topology of $\widehat{A}$ agrees with the $\mathfrak{m}\widehat{A}$-adic one and the completion map $A \to \widehat{A}$ is faithfully flat. These are well-known facts that are contained in almost every introductory reference to commutative algebra. Less well-known is that none of these properties hold for general non-Noetherian rings A - in particular, the completion of a preadic ring need not be adic! The main example to keep in mind is that of $P = k[x_i \mid i \in I]$, the polynomial ring over an infinite set of variables x_i, $i \in I$. In that case, the completion $\widehat{P} = k[[x_i \mid i \in I]]$ is the ring of formal power series in the indeterminates x_i, which, as a topological ring endowed with the limit topology, is not adic.

In this section, we will first collect some general facts about formal power series rings in infinitely many indeterminates and contrast them to the Noetherian case. We will then turn our attention to filtered rings, which can be seen as an algebraic object describing strongly admissible topological rings (see Section 1.1.4). The idea here is to replace the $\mathfrak{a}$-adic filtration

$$\mathfrak{a} \supset \mathfrak{a}^2 \supset \mathfrak{a}^3 \supset \dots$$

by a more general filtration of ideals which gives a fundamental system of neighborhoods of 0. We will introduce some terminology such as the associated graded which we will make use of in later sections and then prove a version of the Weierstrass preparation theorem for strongly admissible rings.

1.2.1 Infinite-variate formal power series rings

Let k be a ring, I any set and $P = k[x_i \mid i \in I]$ the polynomial ring in variables x_i over k. Let $\mathfrak{m} := (x_i \mid i \in I)$ and consider P with the $\mathfrak{m}$-adic topology. Then its completion $\widehat{P}$ is given by the *formal power series ring*

$$\widehat{P} = k[[x_i \mid i \in I]],$$

that is, the elements f of $\widehat{P}$ are formal sums $f = \sum_{n \geq 0} f_n$ where $f_n \in P$ is homogeneous of degree n. We consider $\widehat{P}$ as a topological ring endowed with the limit topology; more precisely, the ideals

$$\widehat{\mathfrak{m}}_n := \ker(\widehat{P} \to P/\mathfrak{m}^n)$$

form a fundamental system of neighborhoods of $\widehat{P}$, with its unique maximal ideal given by $\widehat{\mathfrak{m}} := \widehat{\mathfrak{m}}_1$. Clearly this makes $\widehat{P}$ into an admissible topological ring.

Remark 1.2.1. For $|I| = \infty$ the formal power series ring $\widehat{P} = k[[x_i \mid i \in I]]$ strictly contains the subring

$$\operatorname*{colim}_{J \subset I, |J| < \infty} k[[x_j \mid j \in J]].$$

Note that the latter is also sometimes referred to as the ring of formal power series in variables x_i.

Remark 1.2.2. Let us contrast the above definition of $k[[x_i \mid i \in I]]$ with the ring of formal power series defined in [Bou74, Chapter III, Section 2.11], which we want to briefly recall. For any set I we denote by $\mathbb{N}^{(I)}$ the set of functions $I \to \mathbb{N}$ that take only finitely many non-zero values. Then $\mathbb{N}^{(I)}$ is a monoid, which we identify with the collection of monomials in the variables $\{x_i \mid i \in I\}$ by writing $x^\alpha = \prod_{i \in I,\, \alpha(i) \neq 0} x_i^{\alpha(i)}$ for every $\alpha \in \mathbb{N}^{(I)}$. The k-module $k^{\mathbb{N}^{(I)}}$ can be made into an k-algebra as follows: writing an element $a = (a_\alpha)_{\alpha \in \mathbb{N}^{(I)}}$ as $a = \sum_{\alpha \in \mathbb{N}^{(I)}} a_\alpha x^\alpha$, multiplication is defined via formal extension of $x^\alpha \cdot x^\beta := x^{\alpha + \beta}$. We call $k^{\mathbb{N}^{(I)}}$ the *ring of Bourbaki power series.* Notice that there is a natural inclusion of rings $k[[x_i \mid i \in I]] \subset k^{\mathbb{N}^{(I)}}$. This inclusion is an equality if $|I| < \infty$, and is a strict inclusion if $|I| = \infty$ as in this case $\sum_{i \in I} x_i$ is in $k^{\mathbb{N}^{(I)}}$ but not in $k[[x_i \mid i \in I]]$.

Remark 1.2.3. It is often convenient to expand a formal power series in a subset of the indeterminates, but this becomes delicate in the infinite-variate case. Let I and J be arbitrary sets, and let x_i and y_j be indeterminates indexed by $i \in I$ and $j \in J$, respectively. Dropping for short the index sets from the notation, we have the following inclusions:

$$k[[x_i]] \otimes_k k[[y_j]] \hookrightarrow k[[x_i]][[y_j]] \hookrightarrow k[[x_i, y_j]] \hookrightarrow (k[[x_i]])^{\mathbb{N}^{(J)}}.$$

The first inclusion is always strict, and the other two are equalities if and only if J is finite. For example, if $\mathbb{N} \subset J$ and $x = x_{i_0}$ is one of the indeterminates, then the series $\sum_{n \geq 1} y_n x^n$ belongs to $k[[x_i, y_j]]$ but not to $k[[x_i]][[y_j]]$. Notice that Bourbaki power series are better behaved in this respect, as $k^{\mathbb{N}^{(I \sqcup J)}} = (k^{\mathbb{N}^{(I)}})^{\mathbb{N}^{(J)}} = (k^{\mathbb{N}^{(J)}})^{\mathbb{N}^{(I)}}$.

Let us now compare the following topologies on $\widehat{P} = k[[x_i \mid i \in I]]$:

1. the limit topology given by $\widehat{\mathfrak{m}}_n$,
2. the $\widehat{\mathfrak{m}}$-preadic topology, and
3. the $\mathfrak{m}\widehat{P}$-preadic topology.

It is well-known that for $|I| < \infty$ all these three topologies agree. However, for $|I| = \infty$ the topologies are pairwise distinct. Moreover, $\widehat{P}$ is not complete for both the $\mathfrak{m}$-preadic and the $\mathfrak{m}\widehat{P}$-preadic topology. In particular, $\widehat{P}$ provides a counterexample to the following result from [EGA I]:

Proposition 1.2.4 ([EGA I, 0, (7.2.4)]). *If A is an admissible topological ring and $\mathfrak{a}$ an ideal of A contained in an ideal of definition, then A is separated and complete when considered with the $\mathfrak{a}$-preadic topology.*

Let us mention here that the proof of [EGA I, 0, (7.2.6)], which gives a criterion for an admissible ring to be Noetherian, relies crucially on the above erroneous result. We will give a corrected version for what we call *quasi-adic* rings in Proposition 1.3.16.

Let us now compare the above topologies in the case $|I| = \infty$. We will assume $I = \mathbb{N}$ for convenience's sake, but the arguments clearly work for an arbitrary set I of infinite cardinality. We start by verifying that topology (2) is strictly coarser than (3). Consider $f := \sum_{i \geq 1} x_i^i \in \widehat{\mathfrak{m}}$. Then $f^n \notin \mathfrak{m}\widehat{P}$ for all $n \geq 1$; since otherwise we could find polynomials $a_1, \ldots, a_r \in \mathfrak{m}$ such that

$$f^n = \sum_{i=1}^{r} a_i g_i,$$

with $g_i \in \widehat{P}$. There exists $J \subset \mathbb{N}$ finite such that $a_1, \ldots, a_r \in k[x_j \mid j \in J]$ and setting $x_j = 0$ for all $j \in J$ gives a contradiction. Thus $\mathfrak{m}\widehat{P}$ is not open in the $\widehat{\mathfrak{m}}$-preadic topology. To see that $\widehat{P}$ is not complete with respect to the $\mathfrak{m}\widehat{P}$-preadic topology note that we have exact sequences

$$0 \longrightarrow \widehat{\mathfrak{m}}_n/\mathfrak{m}^n\widehat{P} \longrightarrow \widehat{P}/\mathfrak{m}^n\widehat{P} \longrightarrow P/\mathfrak{m}^n \longrightarrow 0,$$

for $n \geq 1$ which are compatible with transition maps. Now $\mathfrak{m}^n\widehat{P}$ maps onto $\mathfrak{m}^n/\mathfrak{m}^{n+1}$ under the map $\widehat{P} \to P/\mathfrak{m}^{n+1}$. This means that $\mathfrak{m}^n\widehat{P} + \widehat{\mathfrak{m}}_{n+1} =$

$\widehat{\mathfrak{m}}_n$ and thus the inverse system $(\widehat{\mathfrak{m}}_n/\mathfrak{m}^n\widehat{P})_n$ is surjective. Then we can apply Proposition 1.2.8 to get that $\widehat{P} \neq \lim_n \widehat{P}/\mathfrak{m}^n\widehat{P}$, since $f \in \widehat{\mathfrak{m}} \setminus \mathfrak{m}\widehat{P}$.

Finally, to prove that topology (2) is not equivalent to (1) it suffices to show that $\widehat{P}$ is not complete with respect to the $\widehat{\mathfrak{m}}$-preadic topology. The argument is taken from [SP, Tag 05JA] : for $n \geq 1$, we again have exact sequences

$$0 \longrightarrow \widehat{\mathfrak{m}}_n/\widehat{\mathfrak{m}}^n \longrightarrow \widehat{P}/\widehat{\mathfrak{m}}^n \longrightarrow P/\mathfrak{m}^n \longrightarrow 0$$

which are compatible with transition maps. Arguing as above we see that $(\widehat{\mathfrak{m}}_n/\widehat{\mathfrak{m}}^n)_n$ forms a surjective system and by Proposition 1.2.8 it is sufficient to show that $\widehat{\mathfrak{m}}_n$ strictly contains $\widehat{\mathfrak{m}}^n$ for some n. We will do so for $n = 2$ by constructing $f \in \widehat{\mathfrak{m}}_2 \setminus \widehat{\mathfrak{m}}^2$ as a sum

$$f = \sum_{i \geq 1} f_i$$

where $f_i \in k[[x_{n_i}, \ldots, x_{n_{i+1}-1}]]$ are homogeneous of degree d_i and $0 < d_1 < d_2 < \ldots$ and $0 < n_1 < n_2 < \ldots$. If f_i are chosen such that f_i cannot be written as

$$f_i = \sum_{j=1}^{r} g_j h_j,\ r \leq i, \tag{1.2a}$$

with g_j, h_j of order 1, then clearly $f \notin \widehat{\mathfrak{m}}^2$. Thus it suffices to construct the following: for each $i \geq 1$ a homogeneous polynomial $p \in k[x_1, \ldots, x_n]$ of degree $d > 0$ such that p cannot be written as in (1.2a) with $g_i, h_i \in k[[x_1, \ldots, x_n]]$. To that end, we let $n > 2i$ and d prime to the characteristic of the field k. Then let $p = \sum_{j=1}^{n} x_j^d$. Note that the vanishing set of the Jacobian ideal $\mathrm{Jac}(p) = (\frac{\partial p}{\partial x_1}, \ldots, \frac{\partial p}{\partial x_n})$ consists of just the point 0. On the other hand, if p can be written as

$$p = g_1 h_1 + \ldots g_i h_i,$$

then $\mathrm{Jac}(p) \subset (g_1, h_1, \ldots, g_i, h_i)$, which would mean that the vanishing set of $\mathrm{Jac}(p)$ has codimension $2i$ in $\mathbb{A}^n$, a contradiction.

1.2.2 Filtered rings and the associated graded ring

In this section we want to consider the algebraic object representing strongly preadmissible rings from Section 1.1.4, that is, *filtered rings.*

Definition 1.2.5.

1. A *filtered ring* is a ring A together with a descending filtration of ideals of A, that is, ideals $\mathfrak{a}_n \subset A$, $n \in \mathbb{N}_{>0}$, such that $\mathfrak{a}_{n+1} \subset \mathfrak{a}_n$ for all n and $\mathfrak{a}_n \cdot \mathfrak{a}_m \subset \mathfrak{a}_{n+m}$ for all n, m. We denote the datum of A and its filtration by $(A, \mathfrak{a}_n)$.
2. A *filtered module* over a filtered ring $(A, \mathfrak{a}_n)$ is an A-module M together with a descending filtration of submodules of M, that is, submodules $M_n \subset M$, $n \in \mathbb{N}_{>0}$, such that $M_{n+1} \subset M_n$ for all n and $\mathfrak{a}_n \cdot M_m \subset M_{n+m}$ for all n, m. We write (M, M_n) for M and its filtration.
3. A *filtered algebra* over a filtered ring $(A, \mathfrak{a}_n)$ is a filtered ring $(B, \mathfrak{b}_n)$ which is also a filtered A-module. Every filtered ring $(A, \mathfrak{a}_n)$ with A a k-algebra is a filtered algebra over $(k, 0)$.

4. If M is a module over a filtered ring $(A, \mathfrak{a}_n)$, then $(M, \mathfrak{a}_n M)$ is a filtered module. Similarly, if N is a submodule of a filtered module (M, M_n), then $(N, N \cap M_n)$ and $(M/N, N + M_n)$ are filtered modules. We call $N \cap M_n$ the *induced filtration* and $N + M_n$ the *quotient filtration*.

For a filtered ring $(A, \mathfrak{a}_n)$ we set $\mathfrak{a}_0 := A$ and similarly, for a filtered module (M, M_n) set $M_0 := M$.

Definition 1.2.6. A morphism $f : (A, \mathfrak{a}_n) \to (A', \mathfrak{a}'_n)$ between filtered rings is a ring homomorphism $f : A \to A'$ satisfying $f(\mathfrak{a}_n) \subset \mathfrak{a}'_n$ for all $n \in \mathbb{N}$. Morphisms between filtered modules and algebras are defined accordingly.

We can consider each filtered ring $(A, \mathfrak{a}_n)$ as linearly topologized ring by declaring the filtration to be a fundamental system of neighborhoods for 0. Clearly every morphism $(A, \mathfrak{a}_n) \to (A', \mathfrak{a}'_n)$ of filtered rings is continuous in this sense, but not every continuous ring homomorphism $f : A \to A'$ satisfies $f(\mathfrak{a}_n) \subset \mathfrak{a}'_n$.

We call two filtered rings $(A, \mathfrak{a}_n)$ and $(A', \mathfrak{a}'_n)$ *equivalent* if they are isomorphic as topological rings. In particular, two filtrations $\{\mathfrak{a}_n\}$ and $\{\mathfrak{a}'_n\}$ of A are equivalent if and only if, for each $n \in \mathbb{N}$, we have $\mathfrak{a}_n \supset \mathfrak{a}'_{m_n}$ for some m_n and vice versa.

Recall from Definition 1.1.32 that a *strongly preadmissible ring* A is a first-countable linearly topologized ring A which has an ideal of definition. The following proposition tells us that every strongly preadmissible ring can be represented by a filtered ring.

Proposition 1.2.7. *Every filtered ring $(A, \mathfrak{a}_n)$ is strongly preadmissible when considering $\{\mathfrak{a}_n\}$ as a fundamental system of neighborhoods for 0. Conversely, for each strongly preadmissible ring A we may find a sequence of open ideals $\{\mathfrak{a}_n\}$ such that $(A, \mathfrak{a}_n)$ is filtered.*

Proof. For the first assertion note that $\mathfrak{a}_1$ is clearly an ideal of definition. Now assume A is strongly preadmissible. For any ideal of definition $\mathfrak{a}$ we may find a cofinal subsequence for the neighborhood basis of 0 starting with $\mathfrak{a}$. Thus we can assume that

$$\mathfrak{a}'_1 \supset \mathfrak{a}'_2 \supset \mathfrak{a}'_3 \supset \dots$$

is a sequence of open ideals forming a neighborhood basis and $\mathfrak{a}'_1$ is an ideal of definition. By induction assume the first n members $\mathfrak{a}'_1, \dots, \mathfrak{a}'_n$ already satisfy the filtration property, that is,

$$\mathfrak{a}'_{i-j} \cdot \mathfrak{a}'_j \subset \mathfrak{a}'_i \ \forall i \leq n.$$

Now each $\mathfrak{a}'_i$ is clearly an ideal of definition again and thus there exists $c > 0$ such that $(\mathfrak{a}'_n)^c \subset \mathfrak{a}'_{n+1}$. Define a new sequence by setting $\mathfrak{a}_{cn} := \mathfrak{a}'_{n+1}$ and $\mathfrak{a}_i := \mathfrak{a}'_i$ for $i \leq n$. Now fill in the missing elements by recursively defining for $n < i < cn$:

$$\mathfrak{a}_i := \sum_{j=1}^{i-1} \mathfrak{a}_j \cdot \mathfrak{a}_{i-j} + \mathfrak{a}_{cn}.$$

We may think of this construction as "trivially" extending the filtration in $A/\mathfrak{a}_{cn}$. Note that $\mathfrak{a}_i$ is open since $\mathfrak{a}_{cn}$ is. Then it is straightforward to check that the sequence $\mathfrak{a}_1, \dots, \mathfrak{a}_{cn}$ satisfies the filtration property. Since we are only reindexing and adding elements we can exhaust the original sequence of ideals this way. □

We define the *completion* of a filtered ring $(A, \mathfrak{a}_n)$ to be the completion of A as a linearly topologized ring, that is:

$$\widehat{A} = \lim_{n \in \mathbb{N}} A/\mathfrak{a}_n.$$

Note that $\widehat{A}$ is a filtered ring with respect to $\widehat{\mathfrak{a}}_n = \ker(\widehat{A} \to A/\mathfrak{a}_n)$. Similarly, the completion of a filtered $(A, \mathfrak{a}_n)$-module (M, M_n) is

$$\widehat{M} = \lim_{n \in \mathbb{N}} M/M_n$$

and $(\widehat{M}, \widehat{M_n})$ is a filtered module over $(\widehat{A}, \widehat{\mathfrak{a}}_n)$.

We recall that lim is exact on surjective countable inverse systems of abelian groups. One should note that here we are considering inverse systems not as pro-objects, but as functors $\mathbb{N}^{\mathrm{op}} \to \mathrm{Ab}$.

Proposition 1.2.8 ([AM16, Proposition 10.2])**.** *Let $\underline{A}$, $\underline{B}$ and $\underline{C}$ be inverse systems of abelian groups over $\mathbb{N}$ and assume that we have exact sequences*

$$0 \longrightarrow A_n \xrightarrow{f_n} B_n \xrightarrow{g_n} C_n \longrightarrow 0,$$

compatible with the transition maps and assume $\underline{A}$ is surjective (i.e. $A_{n+1} \to A_n$ is surjective for all n). Then the sequence

$$0 \longrightarrow \lim_n A_n \xrightarrow{f} \lim_n B_n \xrightarrow{g} \lim_n C_n \longrightarrow 0$$

is exact.

Remark 1.2.9. This proposition still holds in the case where $\underline{A}$ is only Mittag-Leffler (see the end of Section 1.1.2).

Corollary 1.2.10. *Let (M, M_n) be a filtered $(A, \mathfrak{a}_n)$-module and $N \subset M$ any submodule. Endow N with the induced and M/N with the quotient filtration. Then $\widehat{N}$ is isomorphic to the closure of N inside M and we have $\widehat{M/N} \simeq \widehat{M}/\widehat{N}$.*

Proof. The first assertion follows from the isomorphism

$$N/(N \cap M_n) \simeq (N + M_n)/N.$$

For the second assertion apply the previous proposition to the exact sequence

$$0 \longrightarrow N \cap M_n \longrightarrow M_n \longrightarrow M/(N + M_n) \longrightarrow 0.$$

□

Explicitly considering the filtration has the advantage of being able to work with the relevant associated graded ring:

Definition 1.2.11. If $(A, \mathfrak{a}_n)$ is a filtered ring then the *associated graded ring* $\mathrm{gr}(A)$ is defined as

$$\mathrm{gr}(A) := \bigoplus_{n \in \mathbb{N}} \mathfrak{a}_n / \mathfrak{a}_{n+1}.$$

Similarly, for a filtered module (M, M_n) over $(A, \mathfrak{a}_n)$ its *associated graded module* is defined as

$$\mathrm{gr}(M) := \bigoplus_{n \in \mathbb{N}} M_n / M_{n+1}.$$

Note that $\mathrm{gr}(M)$ is a graded $\mathrm{gr}(A)$-module in the obvious way. Any map f between filtered rings resp. modules gives rise to a map $\mathrm{gr}(f)$ between their associated gradeds. Clearly we have $\mathrm{gr}(\widehat{A}) \simeq \mathrm{gr}(A)$ and $\mathrm{gr}(\widehat{M}) = \mathrm{gr}(M)$.

Example 1.2.12. Let k be a field and $\widehat{P} = k[[x_i \mid\in i \in I]]$. As $\widehat{P}$ is the completion of $P = k[x_i \mid i \in I]$ as a preadic ring with respect to the ideal $\mathfrak{m} = (x_i \mid in \in I)$, we have

$$\mathrm{gr}(\widehat{P}) \simeq \mathrm{gr}(P) \simeq \mathrm{Sym}_k(\mathfrak{m}/\mathfrak{m}^2) \simeq k[x_i \mid i \in I].$$

The usefulness of the associated graded ring in our applications lies in the following fact:

Proposition 1.2.13. *Let $f : (A, \mathfrak{a}_n) \to (A', \mathfrak{a}'_n)$ be a map between filtered rings. Write $\widehat{f} : \widehat{A} \to \widehat{A}'$ for the map between completions and $\mathrm{gr}(f) : \mathrm{gr}(A) \to \mathrm{gr}(A')$ for the map between associated gradeds. Then:*

1. *If $\mathrm{gr}(f)$ is injective, then so is $\widehat{f}$.*
2. *If $\mathrm{gr}(f)$ is surjective, then so is $\widehat{f}$.*

Proof. This is a special case of the equivalent result for filtered groups in [AM16, Lemma 10.23] and follows from Proposition 1.2.8. □

Definition 1.2.14. For $f \in (A, \mathfrak{a}_n)$ we define the *order* of f as $\mathrm{ord}(f) = \sup\{n \mid f \in \mathfrak{a}_n\} \in \mathbb{N} \cup \{\infty\}$. The *initial form* $\mathrm{in}(f) \in \mathrm{gr}(A)$ of f is defined as

$$\mathrm{in}(f) := \begin{cases} \bar{f} \in \mathfrak{a}_d/\mathfrak{a}_{d+1}, & \mathrm{ord}(f) = d \in \mathbb{N} \\ 0, & \mathrm{ord}(f) = \infty. \end{cases}$$

For any ideal $\mathfrak{b} \subset A$ we define the *initial ideal* of $\mathfrak{b}$ as

$$\mathrm{in}(\mathfrak{b}) := (\mathrm{in}(f) \mid f \in \mathfrak{b}) \subset \mathrm{gr}(A).$$

Remark 1.2.15. The function $\mathrm{ord} : A \to \mathbb{N} \cup \{\infty\}$ satisfies the following properties:

1. $\mathrm{ord}(f+g) \geq \min\{\mathrm{ord}(f), \mathrm{ord}(g)\}$,
2. $\mathrm{ord}(fg) \geq \mathrm{ord}(f) + \mathrm{ord}(g)$.

In fact, any function $\nu : A \to \mathbb{N} \cup \{\infty\}$ satisfying these conditions gives rise to a filtration of A by setting $\mathfrak{a}_{\nu,d} := \{a \in A \mid \nu(a) \geq d\}$ for all $d \in \mathbb{N}$.

Proposition 1.2.16 ([Eis95, Proposition 7.12])**.** *Let $(A, \mathfrak{a}_n)$ filtered and assume that A is complete. Let $\mathfrak{b} \subset A$ be an ideal and $f_1, \ldots, f_r \in \mathfrak{b}$ such that $\mathrm{in}(f_1), \ldots, \mathrm{in}(f_r)$ generate $\mathrm{in}(\mathfrak{b})$. Then $f_1, \ldots, f_r$ generate $\mathfrak{b}$ and $\mathfrak{b}$ is closed in A.*

Proof. Only the first assertion is proven explicitly in [Eis95], so let us briefly show that $\mathfrak{b}$ is indeed closed. Let $g \in \bigcap_{n\geq 0} \mathfrak{b} + \mathfrak{a}_n$ and set $d := \max\{\mathrm{ord}(f_i)\}$. Then, for $e > \max\{d, \mathrm{ord}(g)\}$ there exists $g_e \in \mathfrak{b}$ such that $\mathrm{ord}(g - g_e) \geq e$. In particular, $\mathrm{in}(g) \in \mathrm{in}(\mathfrak{b})$. So let $a_{1,e}, \dots, a_{r,e} \in \mathfrak{b}$ such that

$$\mathrm{in}(g) = \sum_{i=1}^{r} \mathrm{in}(a_{i,e}) \,\mathrm{in}(f_i)$$

and $\mathrm{ord}(a_{i,e}) = \deg(\mathrm{in}(a_{i,e})) > e - d > 0$. Replace g with $g' := g - \sum_i a_{i,e} f_i$ and repeat this process to obtain sequences $a_{i,e}$ converging to some a_i and satisfying $g = \sum_{i=1}^{r} a_i f_i$. □

Note that in the special case where $\mathfrak{a}_n$ is the usual filtration of the power series ring $A = k[[x_1, \dots, x_n]]$ this proposition says that any standard basis of $\mathfrak{b}$ already generates $\mathfrak{b}$. We will revisit standard bases in Section 1.5 in the non-Noetherian setting.

We will now give a corrected version of [EGA I, 0, (7.2.4)] in the case of a *finitely generated* ideal. This result was taken from [SP, Tag 09B8] . We first start with a lemma:

Lemma 1.2.17. *Let A be a ring, $\mathfrak{a}$ a finitely generated ideal of A and M an A-module. Then $\widehat{M} = \lim_n M/\mathfrak{a}^n M$ is complete with respect to the $\mathfrak{a}$-preadic topology.*

Proof. Fix $n > 0$. Since $\mathfrak{a}^n$ is finitely generated we have an exact sequence

$$\bigoplus_{i=1}^{r} M \to \mathfrak{a}^n M \to 0.$$

Tensoring with $A/\mathfrak{a}^m$ for $m \geq 0$ gives an exact sequence

$$\bigoplus_{i=1}^{r} M/\mathfrak{a}^m M \to (\mathfrak{a}^n M)/(\mathfrak{a}^m M) \to 0.$$

By Proposition 1.2.8 we have that

$$\bigoplus_{i=1}^{r} \widehat{M} \to \ker(\widehat{M} \to M/\mathfrak{a}^n M) \to 0$$

is exact; recall that finite direct sums and finite direct products of modules agree. Now the image of the first map is just $\mathfrak{a}^n \widehat{M}$ and thus we have $\widehat{M}/\mathfrak{a}^n \widehat{M} \simeq M/\mathfrak{a}^n M$. Taking limits we see that $\widehat{M}$ is complete for the $\mathfrak{a}$-preadic topology. □

Lemma 1.2.18. *Let $\mathfrak{a} \subset A$ be an ideal and M an A-module which is complete for the $\mathfrak{a}$-preadic topology. Then any direct summand of M is complete for the $\mathfrak{a}$-preadic topology.*

Proof. If N is a direct summand of M, then $\mathfrak{a}^n N = \mathfrak{a}^n M \cap N$ for all n. Write $M = N \oplus N'$, then by Proposition 1.2.8 the sequence

$$0 \to \lim_n N/(\mathfrak{a}^n N) \to M \to \lim_n N'/(\mathfrak{a}^n N') \to 0$$

is exact. Let $x \in \lim_n N/(\mathfrak{a}^n N)$ and write $x = x_1 + x_2$ with $x_1 \in N$ and $x_2 \in N'$. As $x_2 = 0$ in $\lim_n N'/(\mathfrak{a}^n N')$, we have $x_2 = 0$ in N' and thus $N \to \lim_n N/(\mathfrak{a}^n N)$ is bijective. □

Now we are ready to prove the following:

Proposition 1.2.19 ([EGA I, 0, (7.2.4)])**.** *If A is a strongly admissible ring and $\mathfrak{a}$ a finitely generated ideal which is contained in an ideal of definition, then A is complete for the $\mathfrak{a}$-preadic topology.*

Proof. Let $\mathfrak{a}'$ be an ideal of definition containing $\mathfrak{a}$. By Proposition 1.2.7 we may choose a filtration $\mathfrak{a}'_1 = \mathfrak{a}' \supset \mathfrak{a}'_2 \supset \dots$ which forms a neighborhood basis of 0. Since $(\mathfrak{a}')^n \supset \mathfrak{a}^n$ we have a sequence

$$A \to A/\mathfrak{a}^n \to A/\mathfrak{a}'_n$$

and taking the limits we obtain a factorization $A \to \lim_n A/\mathfrak{a}^n \to A$ of Id_A. Thus A is a direct summand of $\lim_n A/\mathfrak{a}^n$, which in turn is complete for the $\mathfrak{a}$-preadic topology by Lemma 1.2.17. Then A is complete with respect to the $\mathfrak{a}$-preadic topology by Lemma 1.2.18. □

As Proposition 1.2.19 is the main result used in [EGA I, 0, (7.2.6)] we may give a corrected proof here under the additional assumption that A is *strongly* admissible and $\mathfrak{a}$ is a finitely generated ideal of definition - in the original statement A was just assumed to be admissible and $\mathfrak{a}/\mathfrak{a}^2$ to be finitely generated.

Proposition 1.2.20. *Let A be a strongly admissible ring and $\mathfrak{a}$ an ideal of definition which is finitely generated and such that $A/\mathfrak{a}$ is Noetherian. Then A is Noetherian.*

Proof. By Proposition 1.2.19 the ring A is complete for the $\mathfrak{a}$-preadic topology. Then we may proceed as in [EGA IV_1, 0, (7.2.6)]: disregarding the original topology on A, we consider it with the $\mathfrak{a}$-adic topology and show that $\mathrm{gr}(A)$ is Noetherian, which implies that A is Noetherian by Proposition 1.2.16. To that avail, note that the map

$$\mathrm{Sym}_{A/\mathfrak{a}}(\mathfrak{a}/\mathfrak{a}^2) \to \mathrm{gr}(A)$$

is surjective and since $A/\mathfrak{a}$ is Noetherian, so is $\mathrm{gr}(A)$. □

We have already seen that the completed tensor product $B \widehat{\otimes}_A C$ (see Definition 1.1.20) of admissible rings B and C is again admissible. If B and C are in addition strongly admissible, then $B \otimes_A C$ has again a countable neighborhood basis and thus $B \widehat{\otimes}_A C$ is strongly admissible. For many purposes we would like to choose a particularly well-behaved filtration of $B \otimes_A C$ giving a neighborhood basis, which we will construct now.

Let $(B, \mathfrak{b}_n)$, $(C, \mathfrak{c}_n)$ be filtered algebras over a filtered ring $(A, \mathfrak{a}_n)$. Then $\mathrm{gr}(B)$ and $\mathrm{gr}(C)$ are algebras over $\mathrm{gr}(A)$ and the tensor product $\mathrm{gr}(B) \otimes_{\mathrm{gr}(A)} \mathrm{gr}(C)$ is again graded. The set of homogeneous elements of degree n is given by

$$(\mathrm{gr}(B) \otimes_{\mathrm{gr}(A)} \mathrm{gr}(C))_n := \mathrm{Im}\left(\bigoplus_{d+e=n} \mathfrak{b}_d/\mathfrak{b}_{d+1} \otimes_{A/\mathfrak{a}_1} \mathfrak{c}_e/\mathfrak{c}_{e+1} \right).$$

We endow $B \otimes_A C$ with a filtration by defining

$$\mathfrak{d}_n := \mathrm{Im}\left(\bigoplus_{d+e=n} \mathfrak{b}_d \otimes_A \mathfrak{c}_e \right). \tag{1.2b}$$

Then it is straightforward to check that $(B \otimes_A C, \mathfrak{d}_n)$ is a filtered algebra over $(B, \mathfrak{b}_n)$ and $(C, \mathfrak{c}_n)$.

Proposition 1.2.21. *Consider the filtered algebra $(B \otimes_A C, \mathfrak{d}_n)$ as defined in* (1.2b). *Then:*

1. *The topology on $B \otimes_A C$ induced by the filtration $\mathfrak{d}_n$ agrees with its final topology with respect to the maps $B \to B \otimes_A C$ and $C \to B \otimes_A C$. In particular, the completed tensor product (see Definition 1.1.20) is given by*

$$B \mathbin{\widehat{\otimes}}_A C \simeq \lim_n (B \otimes_A C)/\mathfrak{d}_n.$$

2. *There exists a natural surjection $\mathrm{gr}(B) \otimes_{\mathrm{gr}(A)} \mathrm{gr}(C) \to \mathrm{gr}(B \otimes_A C)$ of graded $\mathrm{gr}(A)$-algebras.*

3. *Let $A = k$ be a field (considered with the trivial filtration). Then the map from (2) is an isomorphism.*

Proof. Let us start by proving (1). It suffices to prove that the filtration $\mathfrak{d}_n$ from (1.2b) is equivalent to the filtration

$$\mathfrak{d}'_{p,q} := \mathrm{Im}(\mathfrak{b}_p \otimes_A C + B \otimes_A \mathfrak{c}_q).$$

This follows from $\mathfrak{d}'_{p,q} \subset \mathfrak{d}_r$ with $r = \min\{p,q\}$, $\mathfrak{d}'_{1,1} = \mathfrak{d}_1$ and $\mathfrak{d}_n \subset \mathfrak{d}'_{m<,m}$ with $m = \left\lfloor \frac{n}{2} \right\rfloor$.

Consider now the map $\varphi : \mathrm{gr}(B) \times \mathrm{gr}(C) \to \mathrm{gr}(B \otimes_A C)$ given by

$$(\overline{b}, \overline{c}) \in \mathfrak{b}_p/\mathfrak{b}_{p+1} \times \mathfrak{c}_q/\mathfrak{c}_{q+1} \mapsto \overline{b \otimes c} \in \mathfrak{d}_{p+q}/\mathfrak{d}_{p+q+1}.$$

It is easy to see that φ is bilinear and $\mathrm{gr}(A)$-balanced. Thus φ induces a map $\widetilde{\varphi} : \mathrm{gr}(B) \otimes_{\mathrm{gr}(A)} \mathrm{gr}(C) \to \mathrm{gr}(B \otimes_A C)$ which is surjective since φ is surjective. This proves (2).

Finally, (3) follows from the corresponding statement for filtered vector spaces. We will repeat the argument here: let $n > 0$ and b_i, $i \in I$, be a k-basis of B such that there exists a chain of subsets

$$I = I_0 \supset I_1 \supset I_2 \supset \ldots \supset I_{n+1}$$

such that b_i, $i \in I_m$, form a k-basis for $\mathfrak{b}_m$. Define elements $c_j \in C$ for a chain of subsets $J_0 \supset \ldots \supset J_{n+1}$ in the same way. A k-basis for

$$\bigoplus_{p+q=n} \mathfrak{b}_p/\mathfrak{b}_{p+1} \otimes_k \mathfrak{c}_q/\mathfrak{c}_{q+1}$$

is given by

$$\overline{b_i} \otimes \overline{c_j},\ (i,j) \in \bigsqcup_{p+q=n} (I_p \setminus I_{p+1}) \times (J_q \setminus J_{q+1}).$$

On the other hand, for $m \leq n+1$ a k-basis for $\mathfrak{d}_m$ is given by

$$\overline{b_i \otimes c_j},\ (i,j) \in \bigsqcup_{m \leq p+q} (I_p \setminus I_{p+1}) \times (J_q \setminus J_{q+1}).$$

Thus the statement follows. $\square$

Example 1.2.22. Note that $\mathrm{gr}(B) \otimes_{\mathrm{gr}(A)} \mathrm{gr}(C) \not\simeq \mathrm{gr}(B \otimes_A C)$ in general. For a simple example, consider $A = k[t]$ with the trivial filtration and $B = C = k[t, x, y]/(tx - x^2, ty - y^4)$. Then one can check that for the associated graded we have

$$\mathrm{gr}(B) \simeq k[t, \bar{x}, \bar{y}]/(t\bar{x}, t\bar{y}^2, \bar{x}^2\bar{y}^2).$$

In particular, the element $f = \bar{x}^2 \otimes \bar{y}^2$ is homogeneous of degree 4. The natural map from (2) of Proposition 1.2.21 maps f to the class of

$$x^2 \otimes y^2 = tx \otimes y^2 = x \otimes ty^2 = x \otimes y^4,$$

which is 0 in $\mathrm{gr}(B \otimes_A B)$.

As a particular case which will be of importance later on, we note that polynomial rings (with coefficients in a filtered ring) behave well when passing to the associated graded.

Lemma 1.2.23. *Let $(A, \mathfrak{a}_n)$ be a filtered ring over any ring k (consider with the trivial filtration). Let $P = k[x_i \mid i \in I]$ endowed with the preadic filtration with respect to $\mathfrak{m} = (x_i \mid i \in I)$, where I is any index set. Consider the ring $A[x_i \mid i \in I] = A \otimes_k P$ with the induced filtration as above. Then the map in* (2) *of Proposition 1.2.21 gives an isomorphism*

$$\mathrm{gr}(A[x_i \mid i \in I]) \simeq \mathrm{gr}(A)[x_i \mid i \in I].$$

We note that this does not immediately follow from Proposition 1.2.21 in the case where A does not contain a field.

Proof. First observe that the induced filtration of $A[x_i \mid i \in I]$ is given by

$$\mathfrak{a}'_n := \sum_{d+e=n} \mathfrak{a}_d \otimes_k \mathfrak{m}^e = \{f = \sum_{\alpha \in \mathbb{N}^{(I)}} f_\alpha x^\alpha \mid f_\alpha \in \mathfrak{a}_{n-|\alpha|}\}.$$

This description gives rise to a map

$$\mathfrak{a}'_n \to \bigoplus_{d+e=n} \mathfrak{a}_d/\mathfrak{a}_{d+1} \otimes_k \mathfrak{m}^e/\mathfrak{m}^{e+1},$$

whose kernel is $\mathfrak{a}'_{n+1}$. Thus we obtain an inverse to the map constructed in the proof of (2) of Proposition 1.2.21. □

As an immediate consequence we will state a rather general version of the formal inverse function theorem. As before, let k be a ring considered with the trivial filtration and $P = k[x_i \mid i \in I]$ and $\widehat{P} = k[[x_i \mid i \in I]]$ with filtrations given by $\mathfrak{m}^n$ and $\widehat{\mathfrak{m}}_n$ as in Section 1.2.1. Given a filtered ring $(S, \mathfrak{n}_n)$, by Example 1.2.12 and Lemma 1.2.23,

$$\mathrm{gr}(S \mathbin{\widehat{\otimes}}_k \widehat{P}) \simeq \mathrm{gr}(S) \otimes_k P.$$

Let us recall here that, as seen in Remark 1.2.3, in general $S \mathbin{\widehat{\otimes}}_k \widehat{P} \not\simeq S[[x_i \mid i \in I]]$. Then we have the following result:

Lemma 1.2.24 (Formal inverse function theorem)**.** *Let $(S, \mathfrak{n}_n)$ be a filtered ring and $\varphi : S \mathbin{\widehat{\otimes}_k} \widehat{P} \to S \mathbin{\widehat{\otimes}_k} \widehat{P}$ a map of filtered S-algebras. Then map $\mathrm{gr}(\varphi)$ on associated graded rings is $\mathrm{gr}(S)$-linear. Write S_0 for the degree 0 elements of $\mathrm{gr}(S)$ and consider the induced map of S_0-modules*

$$D\varphi : \mathfrak{m}'/(\mathfrak{m}')^2 \to \mathfrak{m}'/(\mathfrak{m}')^2,$$

where $\mathfrak{m}'$ is the extension of $\mathfrak{m}$ to $S_0 \otimes_k P$. If $D\varphi$ is an isomorphism, then so is φ.

Proof. By assumption the map $\mathrm{gr}(\varphi)$ is bijective and hence, by Proposition 1.2.13, so is φ. It is then straightforward to check that the inverse φ^{-1} is a map of filtered S-algebras. □

1.2.3 Hensel's lemma and Weierstrass preparation

In this section we want to mention several results usually only proven for complete adic rings that solely rely on completeness with respect to the order and thus readily extend to strongly admissible rings. Let us fix some notation first: we write $(A, \mathfrak{m})$ for a local ring A with maximal ideal $\mathfrak{m}$ and denote its residue field by k. We say that $(A, \mathfrak{m})$ is strongly admissible if A is strongly admissible and $\mathfrak{m}$ is an ideal of definition. If $f \in A[t]$ or $f \in A[[t]]$ than we write $\bar{f}$ for the image of f in $k[t]$ resp. $k[[t]]$ under the natural map.

We start with a very elemental observation:

Lemma 1.2.25. *Let A be a strongly admissible ring and $\mathfrak{a}$ an ideal of definition. For every $x \in \mathfrak{a}$ the element $1 + x$ is invertible.*

Proof. The corresponding geometric series converges. □

A straightforward generalization from the adic case yields that every strongly admissible local $(A, \mathfrak{m})$ is *Henselian*, that is, Hensel's lemma holds for $(A, \mathfrak{m})$.

Proposition 1.2.26 (Hensel's lemma)**.** *Let $(A, \mathfrak{m})$ be a strongly admissible local ring and write $k = A/\mathfrak{m}$. Let $F \in A[t]$ be a monic polynomial and $\bar{F}$ be its image in $k[t]$. If there exists a factorization $\bar{F} = gh$ into coprime monic polynomials $g, h \in k[t]$, then we can find liftings $G, H \in A[t]$ with $\bar{G} = g$, $\bar{H} = h$ and $F = GH$ in $A[t]$.*

Proof. By Proposition 1.2.7 we can choose a filtration of open ideals $\mathfrak{m}_n$ of A which forms a fundamental system of neighborhoods and such that $\mathfrak{m}_1 = \mathfrak{m}$. Then we just repeat the proof of [Mat89, Theorem 8.3] and replace each instance of $a \in \mathfrak{m}^n$ by $a \in \mathfrak{m}_n$. □

Let us now give a proof of the *Weierstrass preparation theorem* for strongly admissible local $(A, \mathfrak{m})$. While the proof is again a straightforward generalization of e.g. [Bou72, VII.8, Proposition 6] we will reproduce it here for the reader, as the Weierstrass decomposition not only is a key step in the proof of the Drinfeld–Grinberg–Kazhdan theorem (see Theorem 3.4.1), but also one of the main obstructions to proving a more global version of said theorem. We first start with an intermediate result.

Proposition 1.2.27. *Let $(A, \mathfrak{m})$ be a strongly admissible local ring and write $k = A/\mathfrak{m}$. If $f \in A[[t]]$ such that $\mathrm{ord}_t(\bar{f}) = s < \infty$, then*

$$A[[t]] \simeq f \cdot A[[t]] \oplus \bigoplus_{i=0}^{s-1} A \cdot t^i.$$

Proof. First identify $M := \bigoplus_i A \cdot t^i$ with the submodule of $A[[t]]$ consisting of polynomials of degree $< s$. We may choose a filtration $\mathfrak{m}_n$ of A forming a neighborhood basis and such that $\mathfrak{m}_1 = \mathfrak{m}$. Endow $A[[t]]$ with the linear topology given by the ideals $\mathfrak{m}_n[[t]]$; it is easy to see that $A[[t]]$ is then complete. Moreover, clearly $M \subset A[[t]]$ is closed.

We essentially repeat the proof in [Bou72] and show first that $fA[[t]] \cap M = (0)$. To that end, write $f = \sum_{i\geq 0} f_i t^i$ and assume that

$$(\sum_{i\geq 0} g_i t^i)(\sum_{i\geq 0} f_i t^i) = \sum_{i=0}^{s-1} r_i t^i. \tag{1.2c}$$

Since the topology on A is separated it is enough to show $g_i \in \mathfrak{m}_n$ for all n. We do this by double induction: suppose $g_i \in \mathfrak{m}_{n-1}$ for all i and $g_i \in \mathfrak{m}_n$ for $i < l$; we need to show $g_l \in \mathfrak{m}_n$. Comparing the coefficients of t^{s+l} in (1.2c) yields

$$(\sum_{i=0}^{l-1} g_i f_{s+l-i}) + g_l f_s + (\sum_{i=l+1}^{s+l} g_i f_{s+l-i}) = 0.$$

By induction, $g_i \in \mathfrak{m}_n$ for $i < l$ and hence so is the left sum; the right sum is in $\mathfrak{m}_n$ since $g_i \in \mathfrak{m}_{n-1}$ and $f_j \in \mathfrak{m}$ for $j < s$. Since f_s is invertible we have $g_l \in \mathfrak{m}_n$.

Now it remains to show that $fA[[t]] + M = A[[t]]$. To start, we set $g := \sum_{i\geq 0} f_{s+i} t^i$; then clearly $f - t^s g \in \mathfrak{m}[[t]]$. Setting $-h := (f - t^s g)g^{-1}$ we have that $h \in \mathfrak{m}[[t]]$. Let $r(t) \in A[[t]]$ and write $r = \sum_{i\geq 0} r_i t^i$. We define iteratively a sequence $q^{(n)} \in A[[t]]$ via

$$q_i^{(0)} := r_{s+i},\ q_i^{(n)} := \sum_{j=0}^{s+i} h_j q_{s+i-j}^{(n-1)}.$$

It is then easy to check that

$$t^s q^{(n)} - hq^{(n-1)} \in M,\ t^s q^{(0)} - r \in M.$$

Since $h \in \mathfrak{m}[[t]]$ we see that $q^{(n)} \in \mathfrak{m}_n[[t]]$. Thus $q := \sum_{n=0}^{\infty} q^{(n)} \in A[[t]]$ and for $m \in \mathbb{N}$ we have

$$t^s(\sum_{n=0}^{m} q^{(n)}) - r - h(\sum_{n=0}^{m-1} q^{(n)}) \in M.$$

Since $M \subset A[[t]]$ is closed, there exists $m \in M$ such that

$$m = r - q(t^s - h) = r - q(t^s + (fg^{-1} - t^s) = r - qg^{-1}f,$$

which proves the claim. $\square$

Definition 1.2.28. A *Weierstrass polynomial* or *distinguished polynomial* of degree s is a monic polynomial $q \in A[t]$ of the form

$$q = t^s + p_{s-1}t^{s-1} + \ldots + p_1 t + p_0$$

and with $p_i \in \mathfrak{m}$.

Corollary 1.2.29 (Weierstrass preparation)**.** *Let $(A, \mathfrak{m})$ be strongly admissible. Let $f \in A[[t]]$ with $\mathrm{ord}_t(\bar{f}) = s < 0$. Then there exists a unique $u \in A[[t]]^*$ and a unique Weierstrass polynomial $q \in A[t]$ of degree s such that $f = uq$. We call $f = uq$ the* Weierstrass decomposition *of f.*

Proof. Apply Proposition 1.2.27 to obtain $t^s = fg' + q'$ for unique $g' \in A[[t]]$ and $q' \in A[t]$ with $\deg(q') < s$. Now consider the above equation in $k[[t]]$ to get $t^s = \bar{f}\bar{g}' + \bar{q}'$. Since $\mathrm{ord}_t(\bar{f}) = s$ we see that q' is a Weierstrass polynomial and that g' is invertible. Setting $g := g'^{-1}$ and $q := t^s + q'$ finishes the proof. □

In Section 3.4 we will revisit the Weierstrass preparation theorem from a more geometric point of view in the setting of arc spaces.

1.3 Quasi-adic rings and the extended Cohen structure theorem

As we have seen in Section 1.2.1, infinite-variate formal power series rings $\widehat{P}$ (and their quotients) are strongly admissible rings which are not adic. However, their topology is not too far from being adic: if $\mathfrak{m}$ denotes the maximal ideal of $\widehat{P}$, then the topological closure $\overline{\mathfrak{m}^n}$ of the n-th power $\mathfrak{m}^n$ is open for all $n \geq 0$. Strongly admissible rings satisfying this property will be called *quasi-adic* and are the main subject of interest of this section. We will prove that every Noetherian quasi-adic ring is already adic, thus showing that the quasi-adic rings are a natural generalization of adic rings to the non-Noetherian case. Moreover, we will prove an extension of the classical *Cohen structure theorem* to quasi-adic rings, where we replace regularity with formal smoothness. Let us remark here that while the definition of quasi-adic rings so far does not seem to be widely considered, the arguments used in this section are either well-known or follow from minor modifications of existing methods.

1.3.1 Quasi-adic rings

Motivated by [EGA I, 0, (7.2.7)] we make the following the definition:

Definition 1.3.1. Let A be a strongly preadmissible topological ring. We say that A is *quasi-preadic* if there exists an ideal of definition $\mathfrak{a}$ of A such that the closure $\overline{\mathfrak{a}^n}$ is open for all $n \in \mathbb{N}$. If A is in addition admissible, i.e. separated and complete, then we say that A is *quasi-adic*.

Example 1.3.2. Every (pre)adic ring is trivially quasi-(pre)adic.

Remark 1.3.3. The requirement that $\mathfrak{a}$ is an ideal of definition is necessary, as even in a preadic ring not every open ideal is an ideal of definition. For example, consider $B = k[x_n \mid n \geq 1]/(x_n^n \mid n \geq 1)$ with k a field and $A = B[t]$ endowed with the (t)-adic topology. Then the maximal ideal $(x_n, t \mid n \geq 1)$ is open, but not an ideal of definition. On the other hand, by [EGA I, 0, (7.1.6)] we have that A has a largest ideal of definition if there exists an ideal of definition $\mathfrak{a}$ of A such that $\mathrm{Nil}(A/\mathfrak{a})$ is nilpotent. Then it is easy to see that every open ideal is already an ideal of definition.

Reformulating Definition 1.3.1 in terms of the underlying inverse system yields:

Lemma 1.3.4. *Each quasi-adic ring is the inverse limit of a surjective system $\underline{A}$ over $\mathbb{N}$ such that $\ker(\pi_{n,n+1}) = \ker(\pi_{1,n+1})^{n+1}$. Conversely, for every such system the limit $A = \lim_i A_i$ is quasi-adic.*

Definition 1.3.5. A *quasi-adic formal scheme* X is a direct limit of schemes X_n, $n \in \mathbb{N}$, such that $X_n \to X_{n+1}$ is a closed immersion and we have

$$\ker(\mathcal{O}_{X_{n+1}} \to \mathcal{O}_{X_n}) = \ker(\mathcal{O}_{X_{n+1}} \to \mathcal{O}_{X_1})^{n+1}.$$

The definition of a quasi-preadic ring is independent of the choice of ideal and choice of filtration, as the following lemma shows:

Lemma 1.3.6. *Let A be a quasi-preadic ring and let $\mathfrak{a}$ be any ideal of definition for A. Then the closure $\overline{\mathfrak{a}^n}$ is open for all $n \geq 1$ and the filtration*

$$\mathfrak{a} \supset \overline{\mathfrak{a}^2} \supset \overline{\mathfrak{a}^3} \supset \dots \tag{1.3a}$$

gives a fundamental system of neighborhoods for 0.

Proof. By assumption there exists an ideal of definition $\mathfrak{m}$ with $\overline{\mathfrak{m}^n}$ open for all n. Then there exists an $e \geq 0$ such that $\mathfrak{m}^e \subset \mathfrak{a}$ and thus $\overline{\mathfrak{a}^n}$ is open for all n. To show that (1.3a) gives a filtration we need to show that $\overline{\mathfrak{a}^n} \cdot \overline{\mathfrak{a}^m} \subset \overline{\mathfrak{a}^{n+m}}$. Since $\overline{\mathfrak{a}^{n+m}}$ is open we have that

$$\overline{\mathfrak{a}^i} = \bigcap_{\mathfrak{b} \subset A \text{ open}} \mathfrak{a}^i + \mathfrak{b} \subset \mathfrak{a}^i + \overline{\mathfrak{a}^{n+m}},\ i \in \mathbb{N}_{>0}.$$

Thus

$$\overline{\mathfrak{a}^n} \cdot \overline{\mathfrak{a}^m} \subset (\mathfrak{a}^n + \overline{\mathfrak{a}^{n+m}}) \cdot (\mathfrak{a}^m + \overline{\mathfrak{a}^{n+m}}) \subset \mathfrak{a}^{n+m} + \overline{\mathfrak{a}^{n+m}} \subset \overline{\mathfrak{a}^{n+m}}.$$

Finally, for any open ideal $\mathfrak{b}$ we have $\mathfrak{a}^d \subset \mathfrak{b}$ for some $d \geq 1$ and since any open ideal is already closed the last assertion follows. $\square$

Corollary 1.3.7. *Let $f : A \to B$ be a continuous ring homomorphism between quasi-preadic rings A and B. Then there exist ideals of definition $\mathfrak{a}$ for A and $\mathfrak{b}$ for B such that $f : (A, \overline{\mathfrak{a}^n}) \to (B, \overline{\mathfrak{b}^n})$ is a morphism of filtered rings.*

Proof. Note that every open ideal contained in an ideal of definition is an ideal of definition itself. Thus for any ideal of definition $\mathfrak{b}$ of B there exists an ideal of definition $\mathfrak{a}$ of A with $f(\mathfrak{a}) \subset \mathfrak{b}$. Then $f^{-1}(\mathfrak{b}^n) \supset \mathfrak{a}^n$ and therefore $f^{-1}(\overline{\mathfrak{b}^n}) \supset \mathfrak{a}^n$. $\square$

Corollary 1.3.8. *A quasi-preadic ring A is preadic if and only if there exists an ideal of definition $\mathfrak{a}$ with $\mathfrak{a}^n = \overline{\mathfrak{a}^n}$ for all $n \geq 1$.*

The following gives a way of determining whether a given filtration is of the form $\overline{\mathfrak{a}^n}$ for some ideal of definition of $\mathfrak{a}$. Of particular importance here is condition (3).

Lemma 1.3.9. *Let $(A, \mathfrak{a}_n)$ be a filtered ring and consider A as a topological ring with respect to the filtration $\{\mathfrak{a}_n\}$. Then the following are equivalent:*

1. $\mathfrak{a}_n \subset \bigcap_{m\geq 0} \mathfrak{a}_1^n + \mathfrak{a}_m$ *for all* $n \geq 1$.
2. $\mathfrak{a}_1^n/\mathfrak{a}_{n+1} \simeq \mathfrak{a}_n/\mathfrak{a}_{n+1}$ *for all* $n \geq 1$.
3. *The natural map*
$$\mathrm{Sym}_{A/\mathfrak{a}_1}(\mathfrak{a}_1/\mathfrak{a}_2) \to \mathrm{gr}(A)$$
is surjective.
4. $\mathrm{gr}(A)$ *is generated by elements of degree* 1 *over* $A/\mathfrak{a}_1$.

In particular, if any of these conditions are met then A is quasi-preadic.

Proof. It is immediate to see that (2), (3) and (4) are equivalent. Assuming (1), we have in particular that $\mathfrak{a}_n \subset \mathfrak{a}_1^n + \mathfrak{a}_{n+1}$, which is clearly equivalent to (2). Now assume (2) and let $f \in \mathfrak{a}_n$. Then there exists $f_{n+1} \in \mathfrak{a}_{n+1}$ such that $f - f_{n+1} \in \mathfrak{a}_1^n$. For f_{n+1} in turn there exists $f_{n+2} \in \mathfrak{a}_{n+2}$ such that $f_{n+1} - f_{n+2} \in \mathfrak{a}_1^{n+1}$. In particular, we have

$$f - f_{n+2} = f - f_{n+1} + f_{n+1} - f_{n+2} \in \mathfrak{a}_1^n.$$

Proceeding inductively, we see that $f \in \bigcap_i \mathfrak{a}_1^n + \mathfrak{a}_i$. □

Proposition 1.3.10. *If A is quasi-preadic, then its completion $\widehat{A}$ is quasi-adic.*

Proof. Choose an ideal of definition $\mathfrak{a}$ of A. By Lemma 1.3.6 the completion of A is given by

$$\widehat{A} = \lim_n A/\overline{\mathfrak{a}^n}.$$

Then the ideals $\widehat{\mathfrak{a}}_n := \ker(\widehat{A} \to A/\overline{\mathfrak{a}^n})$ form a fundamental system of neighborhoods for 0. We have seen before that $\widehat{\mathfrak{a}}_n$ give a filtration. Since the map $\widehat{A} \to A/\overline{\mathfrak{a}^n}$ sends $\widehat{\mathfrak{a}}_1$ to $\mathfrak{a}/\overline{\mathfrak{a}^n}$, we have that

$$\widehat{\mathfrak{a}}_1^n/\widehat{\mathfrak{a}}_{n+1} \simeq \mathfrak{a}^n/\overline{\mathfrak{a}^{n+1}}.$$

By Lemma 1.3.9 the right hand side is equal to $\overline{\mathfrak{a}^n}/\overline{\mathfrak{a}^{n+1}}$, which in turn is isomorphic to $\widehat{\mathfrak{a}}_n/\widehat{\mathfrak{a}}_{n+1}$. Applying Lemma 1.3.9 again gives the result. □

Example 1.3.11. As every preadic ring is quasi-preadic, the formal power series ring $k[[x_i \mid i \in I]]$ with I infinite is quasi-adic but not adic.

Example 1.3.12. Here is an example of a strongly admissible ring that is not quasi-adic. Let $(A, \mathfrak{a})$ be a domain that is quasi-adic but not adic, e.g. $A = k[[x_n \mid n \in \mathbb{N}]]$. Let $A\{t\}$ denote the ring of *restricted power series* in A, that is,

$$A\{t\} = \{f(t) \in A[[t]] \mid f(t) = \sum_{i\geq 0} a_i t^i,\ a_i \in \overline{\mathfrak{a}^i}\}.$$

Now endow $A\{t\}$ with the linear topology given by

$$\mathfrak{m}_i := \{f(t) \in A\{t\} \mid \mathrm{ord}_t(f) \geq i\}.$$

It is easy to check that this makes $A\{t\}$ into a strongly admissible ring. Since A is not adic, there exists an $m > 0$ such that for each $n \geq m$ we can find $a_n \in \overline{\mathfrak{a}^n}$ and $a_n \notin \mathfrak{a}^m$. Then $a_n t^n \in \mathfrak{m}_n$ but $a_n t^n \notin \mathfrak{m}_1^m + \mathfrak{m}_{n+1}$. In particular, $\overline{\mathfrak{m}_1^m} = \bigcap_{l \geq 0} \mathfrak{m}_1^m + \mathfrak{m}_l$ does not contain $\mathfrak{m}_n$ for all $n \geq 0$ and is thus not open.

Lemma 1.3.13. *Let A be a quasi-preadic ring and $\mathfrak{b} \subset A$ any ideal. Then the quotient $A/\mathfrak{b}$ endowed with the quotient topology is again quasi-preadic. If A is in addition quasi-adic and $\mathfrak{b} \subset A$ closed, then $A/\mathfrak{b}$ is quasi-adic.*

Proof. Let $\mathfrak{a}$ be an ideal of definition and consider $(A, \overline{\mathfrak{a}^n})$ as a filtered ring. Then $A/\mathfrak{b}$ is filtered with respect to the quotient filtration $\mathfrak{b} + \overline{\mathfrak{a}^n}$. Clearly

$$(\mathfrak{a}^n + \mathfrak{b})/(\mathfrak{b} + \overline{\mathfrak{a}^{n+1}}) \simeq (\overline{\mathfrak{a}^n} + \mathfrak{b})/(\mathfrak{b} + \overline{\mathfrak{a}^{n+1}})$$

and by Lemma 1.3.9 $A/\mathfrak{b}$ is quasi-preadic. The second assertion follows from Corollary 1.2.10. ☐

Proposition 1.3.14. *Let $f : B \to A$ and $g : C \to A$ be morphisms of quasi-preadic rings. Then the tensor product $B \otimes_A C$ is again quasi-preadic and the completed tensor product $B \widehat{\otimes}_A C$ is quasi-adic.*

Proof. Let $\mathfrak{b}$ and $\mathfrak{c}$ be ideals of definition for B respectively C. Denote by $e : B \to B \otimes_A C$ and $f : C \to B \otimes_A C$ the natural map. Since $B \otimes_A C$ carries the final topology with respect to e and f, the ideal $\mathfrak{d}$ generated by $e(\mathfrak{b})$ and $f(\mathfrak{c})$ is open. Then $e^{-1}(\overline{\mathfrak{d}^n}) \supset \overline{\mathfrak{b}^n}$ and $f^{-1}(\overline{\mathfrak{d}^n}) \supset \overline{\mathfrak{c}^n}$, so $\overline{\mathfrak{d}^n}$ is open. If $\mathfrak{d}'$ is any other open ideal, then $e^{-1}(\mathfrak{d}') \supset \mathfrak{b}^d$ and $f^{-1}(\mathfrak{d}') \supset \mathfrak{c}^e$ for some $c, e > 0$. Then $\mathfrak{d}' \supset \mathfrak{d}^{d+e}$ and $\mathfrak{d}$ is an ideal of definition. ☐

The following proposition shows that quasi-adic rings are a direct generalization of adic rings to the non-Noetherian case. The statement and proof are taken (with minor modifications) from [SP, Tag 09B8] .

Proposition 1.3.15. *Let A be a quasi-adic ring and $\mathfrak{a}$ be a finitely generated ideal of definition. Then A is adic.*

Proof. Set $A_n := A/\overline{\mathfrak{a}^n}$ and $K_n := \ker(A \to A_n)$. This gives an inverse system of short exact sequences

$$0 \to K_n \to A \to A_n \to 0.$$

Since A is quasi-adic we have that $\ker(A_{n+1} \to A_n) = \mathfrak{a}^n A_{n+1}$. By the snake lemma, we see that $\mathrm{coker}(K_{n+1} \to K_n) \simeq \mathfrak{a}^n A_{n+1}$. This implies that the map

$$K_{n+1}/(K_{n+1} \cap \mathfrak{a}^{n+1}A) \to K_n/(K_n \cap \mathfrak{a}^n A) \tag{1.3b}$$

is surjective. Taking the limit of the inverse system of short exact sequences

$$0 \to K_n/(K_n \cap \mathfrak{a}^n) \to A/\mathfrak{a}^n \to A_n \to 0$$

we obtain a short exact sequence

$$0 \to \lim_n K_n/(K_n \cap \mathfrak{a}^n A) \to \lim_n A/\mathfrak{a}^n \to A \to 0.$$

By Proposition 1.2.19 we have $\lim_n A/\mathfrak{a}^n \simeq A$, thus $\lim K_n/(K_n \cap \mathfrak{a}^n A) = 0$ and since the transition maps (1.3b) are surjective we see that $K_n/(K_n \cap \mathfrak{a}^n A) = 0$ for all n. Therefore $A/\mathfrak{a}^n = A_n$ and $\mathfrak{a}^n = \overline{\mathfrak{a}^n}$. $\square$

For quasi-adic rings we may also adapt the statement of [EGA I, 0, (7.2.6)] to obtain the following result. Note the main difference to Proposition 1.2.20 is that we do not require $\mathfrak{a}$ itself to be finitely generated, which makes the assumptions slightly more general.

Proposition 1.3.16. *Let A be a quasi-adic ring and $\mathfrak{a}$ an ideal of definition. Then A is Noetherian if and only if $A/\mathfrak{a}$ is Noetherian and $\mathfrak{a}/\overline{\mathfrak{a}^2}$ is finitely generated over $A/\mathfrak{a}$. In particular, A is adic.*

Proof. The "if"-direction is clear. For the other implication let $x_1, \dots, x_r$ be generators for $\mathfrak{a}/\overline{\mathfrak{a}^2}$ over $A/\mathfrak{a}$. Then the map from Lemma 1.3.9

$$(A/\mathfrak{a})[x_1, \dots, x_r] \simeq \mathrm{Sym}_{A/\mathfrak{a}}(\mathfrak{a}/\overline{\mathfrak{a}^2}) \to \mathrm{gr}(A)$$

is surjective and thus $\mathrm{gr}(A)$ is Noetherian. Thus for any ideal $\mathfrak{b} \subset A$ its initial ideal $\mathrm{in}(\mathfrak{b})$ is finitely generated. But then $\mathfrak{b}$ is already closed and finitely generated by Proposition 1.2.16. In particular, $\mathfrak{a}^n = \overline{\mathfrak{a}^n}$. $\square$

1.3.2 The extended Cohen structure theorem

Let $(A, \mathfrak{m})$ be a local ring, then we say that A is *local quasi-(pre)adic* if A is quasi-(pre)adic and $\mathfrak{m}$ an ideal of definition. Note that any continuous map between local quasi-(pre)adic rings is already local. In this section we want to provide an extension of the structure theorem for Noetherian local adic rings, as proven first by Cohen in [Coh46], to non-Noetherian local quasi-adic rings. There are various results proven in [Coh46] which are referred to as "Cohen structure theorem", so let us specify the version we aim to generalize:

Theorem 1.3.17 (Cohen structure theorem)**.** *Let $(A, \mathfrak{m})$ be a Noetherian local ring which is complete for the $\mathfrak{m}$-adic topology.*

1. *There exists a surjection*

$$\Lambda[[x_1, \dots, x_n]] \to A,$$

where Λ is a coefficient ring for A (see Definition 1.3.39).

2. *If in addition A contains a field k (i.e. A is* equicharacteristic*), then A is regular if and only if $A \simeq K[[x_1, \dots, x_n]]$, where $K = A/\mathfrak{m}$.*

To give a basic outline of this section: we will first prove the analogue of assertion (2) for quasi-adic rings and then later discuss the existence of coefficient rings for local quasi-adic rings of mixed characteristic. Most of the modifications needed are rather minor in nature. In particular, the hardest part of (1) is establishing the existence of so-called *Cohen rings*, which is done for example in [EGA IV$_1$, 0, (19.7)] and will not be proven again here. Similarly, the most important step in proving (2) is the fact that any field extension is separable if and only if it is formally smooth (see Theorem 1.3.23); we refer the reader to [EGA IV$_1$, 0, (19.6.1)]. Both of these statements can be used without any modification for our proof.

The first obstacle to generalizing Theorem 1.3.17 to non-Noetherian rings is that the regularity condition does not make sense for rings of infinite Krull dimension. We will thus replace it with the notion of *formal smoothness* for topological rings. In fact, this is esentially an intermediate step in proving Theorem 1.3.17 in [EGA IV_1]. Let us recall the definition:

Definition 1.3.18 ([EGA IV_1, 0, (19.3.1)])**.** Let k be a topological ring and A a topological k-algebra. Then A is called *formally smooth* over k if for every commutative square

$$\begin{tikzcd} k \arrow[r] \arrow[d] & C \arrow[d] \\ A \arrow[r] \arrow[ur, dashed] & C/\mathfrak{c}, \end{tikzcd}$$

where C is a discrete topological ring and $\mathfrak{c} \subset C$ an ideal with $\mathfrak{c}^2 = 0$, there exists a morphism $A \to C$ of topological rings making the diagram commute.

Remark 1.3.19. Consider the embedding Ring $\to$ TRing which endows each ring with the discrete topology. Via this embedding Definition 1.3.18 reduces to the (usual) notion of formally smooth ring homomorphisms.

Lemma 1.3.20 ([EGA IV_1, 0, (19.3.5)(ii)])**.** *If A is a formally smooth topological k-algebra and $k \to k'$ a morphism of topological rings, then $A \otimes_k k'$ is formally smooth over k'.*

A quick word of warning: any local ring $(A, \mathfrak{m})$ essentially of finite type and formally smooth over a field k is already regular. The converse is not true in general; one needs to replace A regular with A being *geometrically regular*, i.e. $A \otimes_k \overline{k}$ is regular, where $\overline{k}$ is the algebraic closure of k. However, this distinction is not too relevant here as any power series ring over a field is always formally smooth with respect to its inverse limit topology:

Lemma 1.3.21. *Let k be a topological ring.*

1. *Let $A \in$ LTRing be formally smooth over k. Then the completion $\widehat{A}$ of A is formally smooth over k as well.*

2. *Consider $\mathbb{Z}$ with the discrete topology and $\mathbb{Z}[x_i \mid i \in I]$ with the preadic topology with respect to the ideal $(x_i \mid i \in I)$. Then $k[x_i \mid i \in I] = k \otimes_{\mathbb{Z}} \mathbb{Z}[x_i \mid i \in I]$ is formally smooth over k.*

Proof. To see (1), given a commutative square

$$\begin{tikzcd} k \arrow[r] \arrow[d] & C \arrow[d] \\ \widehat{A} \arrow[r] & C/\mathfrak{c} \end{tikzcd}$$

as in Definition 1.3.18, we see that there exists a morphism $A \to C$, which extends to a diagonal arrow $\widehat{A} \to C$ since C is discrete (and thus complete).

Now, by Lemma 1.3.20, to prove (2) it is sufficient to prove that $B = \mathbb{Z}[x_i \mid i \in I]$ is formally smooth over $\mathbb{Z}$. Write $\mathfrak{m} = (x_i \mid i \in I)$ and consider a commutative square

$$\begin{tikzcd} k \arrow[r] \arrow[d] & C \arrow[d] \\ B \arrow[r] & C/\mathfrak{c} \end{tikzcd}$$

with C discrete and $\mathfrak{c}^2 = 0$. For each $x_i \in I$ choose a lifting in C of the image of x_i in $C/\mathfrak{c}$ to obtain a map $B \to C$. Since for some $n > 0$ the image of $\mathfrak{m}^n$ in $C/\mathfrak{c}$ is 0, we get that the image of $\mathfrak{m}^{2n}$ is 0 in C and thus the map $B \to C$ is continuous. □

We start with the following definition:

Definition 1.3.22. Let $(A, \mathfrak{m})$ be a local quasi-adic ring. A subfield $K \subset A$ is called a *coefficient field* for A if it maps isomorphically onto $A/\mathfrak{m}$. If A is in addition a topological algebra over a topological ring k, then a coefficient field K over k is a coefficient field containing the image of k in A.

If $(A, \mathfrak{m})$ local quasi-preadic, then a *formal coefficient field* for A is a coefficient field K for $\widehat{A}$.

By definition every coefficient field is considered with the discrete topology. The first main result is the following:

Theorem 1.3.23 ([EGA IV$_1$, 0, (19.6.1)])**.** *Let $k \subset K$ be an extension of fields. Then $k \subset K$ is separable if and only if K is formally smooth over k (as discrete topological fields).*

Corollary 1.3.24. *Let $(A, \mathfrak{m})$ be a local quasi-adic ring.*

1. *If A is a topological algebra over a discrete ring k. If $k \to A/\mathfrak{m}$ is formally smooth, then there exists a coefficient field K for A over k.*

2. *If A is equicharacteristic, then there exists a coefficient field K for A.*

Proof. Write $K' = A/\mathfrak{m}$. If K' is formally smooth over k, then for every $n \in \mathbb{N}$ there exists a diagonal arrow to the commutative square

$$\begin{tikzcd} k \arrow[r] \arrow[d] & A/\overline{\mathfrak{m}^{n+1}} \arrow[d] \\ K' \arrow[r] \arrow[ur, dashed] & A/\overline{\mathfrak{m}^{n}}. \end{tikzcd}$$

Note that $\overline{\mathfrak{m}^n}^2 = 0$ in $A/\overline{\mathfrak{m}^{n+1}}$. Thus we obtain a map $K' \to A$ whose image K is a coefficient field for A.

Now assume that A is equicharacteristic. Let p the characteristic of $K' := A/\mathfrak{m}$. Then A contains the prime field k of characteristic p and since k is perfect the composition $k \to A \to K'$ is a separable field extension. By Theorem 1.3.23 K' is a formally smooth k-algebra and we can apply part (1) to see the claim. □

For a finite-variate formal power series ring $A = k[[x_1, \ldots, x_n]]$ the variables x_i can be characterized, up to isomorphism, by either saying that they form a minimal set of generators for $\mathfrak{m} = (x_1, \ldots, x_n)$, or that they form a maximal regular sequence. This characterization no longer works in the infinite-variate case, as the maximal ideal of $k[[x_i \mid i \in I]]$ is not generated by the elements x_i (see Section 1.2.1). This leads us to consider the following definition:

Definition 1.3.25. Let $(A, \mathfrak{m})$ be a local quasi-preadic ring. Then the $A/\mathfrak{m}$-vector space $\mathfrak{m}/\overline{\mathfrak{m}^2}$ is called the *continuous Zariski cotangent space.* A collection of elements $x_i \in \mathfrak{m}$, $i \in I$, whose images in $\mathfrak{m}/\overline{\mathfrak{m}^2}$ form a basis are called *formal coordinates.*

Example 1.3.26. For any local adic ring $(A, \mathfrak{m})$ the continuous Zariski cotangent space is just the usual cotangent space. However, if I is infinite, then the continuous Zariski cotangent space of $A = k[[x_i \mid i \in I]]$ is isomorphic to $(x_i \mid i \in I)/(x_i \mid i \in I)^2$ and a proper quotient of the (usual) cotangent space of A.

Remark 1.3.27. If $(A, \mathfrak{m})$ is a local quasi-adic ring admitting a coefficient field K, then the continuous cotangent space of A is isomorphic to $\widehat{\Omega}_{A/K} \otimes_{\widehat{A}} K$, where

$$\widehat{\Omega}_{A/K} := \varprojlim_n \Omega_{(A/\overline{\mathfrak{m}^n})/K}$$

is defined as in [EGA IV$_1$, 0, (20.7)].

Definition 1.3.28. For any local quasi-preadic ring $(A, \mathfrak{m})$ the *embedding dimension* of A is defined to be the dimension of the continuous Zariski cotangent space, that is,

$$\operatorname{edim}(A) := \dim_{A/\mathfrak{m}}(\mathfrak{m}/\overline{\mathfrak{m}^2}) \in \mathbb{N} \cup \{\infty\}.$$

Remark 1.3.29. Let us remark that, for A local quasi-adic, this definition agrees with the usual definition of the embedding dimension of A as a local ring. More precisely, if $\operatorname{edim}(A) < \infty$, then by Proposition 1.3.16 A is adic and thus its continuous cotangent space agrees with its (usual) Zariski cotangent space. If $\operatorname{edim}(A) = \infty$, then the continuous cotangent space is a proper subspace of the usual cotangent space and thus the (usual) embedding dimension is infinite as well.

In particular, Definition 1.3.25 allows us to give a well-behaved notion of cotangent map for morphisms between local quasi-adic rings.

Definition 1.3.30. Let $f : (A, \mathfrak{m}) \to (B, \mathfrak{n})$ be a morphism between local quasi-adic rings; write $K := A/\mathfrak{m}$ and $L := B/\mathfrak{n}$. Then f induces an L-linear map

$$Df : \mathfrak{m}/\overline{\mathfrak{m}^2} \otimes_K L \to \mathfrak{n}/\overline{\mathfrak{n}^2},$$

which we call the *continuous cotangent map* of f.

Applying Lemma 1.2.24 we obtain the usual formal inverse function theorem.

Corollary 1.3.31 (Formal inverse function theorem)**.** *Let k be a field and $\widehat{P} = k[[x_i \mid i \in I]]$ for some index set I. Then any continuous endomorphism $f : \widehat{P} \to \widehat{P}$ is an isomorphism if and only if its continuous cotangent map Df is.*

Definition 1.3.32. Let $(A, \mathfrak{m})$ be a local quasi-adic ring. A *formal embedding* of A is a surjective morphism $\widehat{P} \to A$ of topological rings, where $\widehat{P} = K[[x_i \mid i \in I]]$

is a formal power series ring over K. If the continuous cotangent map Df of $\widehat{P} \to A$ is an isomorphism, then we call it an *efficient formal embedding.*

If A is in addition a topological k-algebra, then a formal embedding $\widehat{P} \to A$ over k is a formal embedding that is a morphism of topological k-algebras.

We are now ready to prove the following theorem, which serves as an extension of part of the classical Cohen structure theorem to local quasi-adic rings. One thing of note is that in assertion (1), a coefficient field might not exist.

Theorem 1.3.33 (Extended Cohen structure theorem)**.** *Let $(A, \mathfrak{m})$ be a local quasi-adic ring which is a topological algebra over a discrete ring k.*

1. *Every choice of coefficient field K over k and formal coordinates $a_i \in A$, $i \in I$, determine an efficient formal embedding $\widehat{P} = K[[x_i \mid i \in I]] \to A$ over k given by $x_i \mapsto a_i$. Conversely, every efficient formal embedding is of this form.*
2. *Assume that $k \to A/\mathfrak{m}$ is formally smooth. Then an efficient formal embedding $f : \widehat{P} \to A$ over k is an isomorphism if and only if A is formally smooth over k.*
3. *If $k \to A/\mathfrak{m}$ is formally smooth and $f : \widehat{P} \to A$ and $f' : \widehat{P}' \to A$ are efficient formal embeddings of A over k, then there exists an isomorphism $g : \widehat{P}' \to \widehat{P}$ of topological rings such that $f' = f \circ g$.*

Example 1.3.34. In assertion (2) the assumption that $k \to A/\mathfrak{m}$ is formally smooth is necessary. Consider $A = \mathbb{F}_p(t)[x]/(x^p - t)$ over $k = \mathbb{F}_p(t)$; in other words, the field extension obtained by adjoining a p-th root of t to $\mathbb{F}_p(t)$. Then A is not formally smooth over k as it is not separable, but the identity on A gives an efficient formal embedding of A.

Assuming that A is in addition equicharacteristic and choosing k in Theorem 1.3.33 to be the prime field contained in A we obtain as a consequence:

Corollary 1.3.35. *Let $(A, \mathfrak{m})$ be a local quasi-adic ring which is equicharacteristic. Then there exists an efficient formal embedding $f : \widehat{P} \to A$. Moreover, A is isomorphic to a formal power series ring over its residue field $A/\mathfrak{m}$ if and only if A is formally smooth over its prime field k.*

Proof. By Corollary 1.3.24 there exists a coefficient field K for A over its prime field k and by (1) of Theorem 1.3.33 any choice of formal coordinates give an efficient formal embedding. On the other hand, every coefficient field K for A clearly contain k. Since k is perfect, we have that $k \to A/\mathfrak{m}$ is separable and thus formally smooth. Therefore the last assertion follows from (2) of Theorem 1.3.33. □

Proof of Theorem 1.3.33. The proof is a slight variation of that of [EGA IV$_1$, 0, (19.5.4.2)]. Write $A_n := A/\overline{\mathfrak{m}^n}$ and $S := \mathrm{Sym}_K(\mathfrak{m}/\overline{\mathfrak{m}^2})$ as well as $\mathfrak{n} := \ker(S \to K)$ and $S_n := S/\mathfrak{n}^n$.

We start by proving (1). A choice of a coefficient field K of A over k corresponds to a choice of a k-linear section $K \xrightarrow{s} A \to K$. Similarly, a choice of formal coordinates $a_i \in A$, $i \in I$, corresponds to choosing a K-linear section $\mathfrak{m}/\overline{\mathfrak{m}^2} \xrightarrow{s'} \mathfrak{m} \to \mathfrak{m}/\overline{\mathfrak{m}^2}$. The datum of s and s' determine a unique map of filtered rings $v : (S, \mathfrak{n}^n) \to (A, \overline{\mathfrak{m}^n})$. In particular, after identifying S with its associated graded $\mathrm{gr}(S)$, the map $\mathrm{gr}(v)$ is just the natural map

$$S = \mathrm{Sym}_K(\mathfrak{m}/\overline{\mathfrak{m}^2}) \to \mathrm{gr}(A),$$

which is surjective by Lemma 1.3.9. Since $\widehat{S} \simeq \widehat{P} = K[[x_i \mid i \in I]]$ with x_i mapping to $\bar{a}_i \in \mathfrak{m}/\overline{\mathfrak{m}^2}$, the map $\widehat{v} : \widehat{S} \to A$ gives an efficient formal embedding by Proposition 1.2.13. Conversely, given any efficient formal embedding $f : \widehat{P} \to A$ clearly $f(K)$ is a coefficient field of A and the image of x_i in $\mathfrak{m}/\overline{\mathfrak{m}^2}$ form a basis.

To prove (2), we assume first that A is formally smooth over k. Consider the morphism $v_n : S_n \to A_n$ induced by v, which is surjective and has nilpotent kernel. Thus, since A is formally smooth over k, there exists a diagonal arrow w_n for the commutative square

$$\begin{tikzcd} k \arrow[r] \arrow[d] & S_n \arrow[d] \\ A \arrow[r] \arrow[ur, dashed, "w_n"'] & A_n \end{tikzcd}$$

which is continuous. Taking inverse limits we obtain a factorization $A \xrightarrow{w} S \xrightarrow{v} A$ of id_A. Denoting by $\mathrm{gr}^d(w) : \mathrm{gr}^d(A) \to \mathrm{gr}^d(S)$ the map between homogeneous elements of degree d of the associated graded rings, we see that $\mathrm{gr}^0(w) = \mathrm{id}_K$ and $\mathrm{gr}^1(w) = \mathrm{id}_{\mathfrak{m}/\overline{\mathfrak{m}^2}}$. Thus the composition $\mathrm{gr}(S) \xrightarrow{\mathrm{gr}(v)} \mathrm{gr}(A) \xrightarrow{\mathrm{gr}(w)} \mathrm{gr}(S)$ is the identity on $\mathrm{gr}(S)$, which in turn implies that $\mathrm{gr}(v)$ is injective. Therefore the efficient formal embedding $\widehat{S} \to A$ is an isomorphism.

Finally, to prove (3), let $f : \widehat{P} \to A$ and $f' : \widehat{P}' \to A$ be efficient formal embeddings corresponding to the choice of coefficient fields K and K' as well as formal coordinates a_i, $i \in I$ and a'_i, $i \in I$. Denote by $\mathfrak{n}$ the maximal ideal of $\widehat{P}$. For each $n \in \mathbb{N}$ consider the commutative square

$$\begin{tikzcd} k \arrow[r] \arrow[d] & \widehat{P}/\overline{\mathfrak{n}^n} \arrow[d, "f_n"] \\ K' \arrow[r] & A/\overline{\mathfrak{m}^n}. \end{tikzcd}$$

Since $\ker(f_n)$ is nilpotent and $k \to K'$ is formally smooth, for each n there exists a diagonal map $K' \to \widehat{P}/\overline{\mathfrak{n}^n}$ and taking the limit we obtain a map $K' \to \widehat{P}$ which gives an isomorphism $K' \to K$. As f is surjective, there are elements $g_i \in \widehat{P}$ with $f(g_i) = a'_i \in A$. The morphism $g : \widehat{P}' \to \widehat{P}$ is then obtained as the completion of the morphism $P' = K'[x_i \mid i \in I] \to \widehat{P}$ given by $x_i \mapsto g_i$. $\square$

Remark 1.3.36. Part (2) of Theorem 1.3.33 can also be derived directly as a consequence of [EGA IV_1, 0, (19.5.3)]. To see this, apply part (ii) of the latter to obtain that the map

$$\varphi_n : \mathrm{Sym}^n_{A/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2) \to \mathfrak{m}^n/\mathfrak{m}^{n+1}$$

is what is called a *bimorphisme formel* in [EGA IV_1, 0, (19.1.2)], that is, the completion of φ_n is an isomorphism. As $\mathfrak{m}^n/\mathfrak{m}^{n+1}$ carries the induced topology, its completion is given by $\mathfrak{m}^n/\overline{\mathfrak{m}^{n+1}}$ and we can apply Lemma 1.3.9 to deduce part (2) of Theorem 1.3.33. As the proof of [EGA IV_1, 0, (19.5.3)] is rather involved and the level of generality not necessary for our purposes, we decide to include a far shorter and self-contained proof here.

Remark 1.3.37. By the same argument of the proof of part (3) of Theorem 1.3.33, one can see that given any two formal embeddings $\tau\colon \widehat{P} \to A$ and $\tau'\colon \widehat{P}' \to A$ (not necessarily efficient) there is always a map $\varphi\colon \widehat{P}' \to \widehat{P}$ such that $\tau' = \tau \circ \varphi$, and if τ efficient then φ is surjective.

An immediate consequence of Theorem 1.3.33 is that the embedding dimension of A can be computed by via its formal embeddings as follows:

Corollary 1.3.38. *Let $(A, \mathfrak{m})$ be a local quasi-adic ring which is equicharacteristic. Then*

$$\operatorname{edim}(A) = \min_f \dim \widehat{P}$$

where the minimum is taken over all choices of formal embeddings $f\colon \widehat{P} \to A$ and is achieved whenever f is an efficient formal embedding.

Proof. If $f : \widehat{P} \to A$ is any formal embedding, then the map on associated graded rings $\operatorname{gr}(f)$ is surjective by Theorem 1.3.33. In particular the continuous cotangent map Df is surjective. Therefore $\operatorname{edim}(A) \leq \operatorname{edim}(\widehat{P}) = \dim \widehat{P}$. If f is efficient, then Df is an isomorphism and equality holds. $\square$

Let us now briefly turn our attention to the mixed characteristic case, where we essentially follow [SP, Tag 0323] . As the existence of a coefficient field is no longer guaranteed for quasi-adic rings of mixed characteristic, we need to allow more general *coefficient rings* to obtain an analogue to the formal embeddings established in Corollary 1.3.35. The required properties are often specified as follows:

Definition 1.3.39. If $(A, \mathfrak{m})$ is a local quasi-adic ring, then a *coefficient ring* for A is a continuous injection $(\Lambda, p\Lambda) \to (A, \mathfrak{m})$ with Λ local with maximal ideal $p\Lambda$ and p the characteristic of $A/\mathfrak{m}$ such that Λ is complete with respect to the $p\Lambda$-adic topology and $\Lambda/p\Lambda \simeq A/\mathfrak{m}$.

Note that the map $\Lambda \to A$ being continuous is equivalent to $\Lambda \cap \overline{\mathfrak{m}^n} = p^n\Lambda$. Definition 1.3.39 can be broken down to mean the following:

1. If the characteristic p of $A/\mathfrak{m}$ is 0 or $pA = 0$, then any coefficient ring Λ for A is just a field (endowed with the discrete topology). This is the equicharacteristic case we have already discussed.

2. If $p > 0$ and p is not nilpotent in A, then a coefficient ring Λ is a complete discrete valuation ring with residue field $A/\mathfrak{m}$ and uniformizer p.

3. If $p > 0$ and $p^n = 0$ in A, then a coefficient ring Λ is an Artinian local ring with residue field $A/\mathfrak{m}$ and maximal ideal generated by p.

We will now prove that every local quasi-adic $(A, \mathfrak{m})$ has a coefficient ring. As the proof is practically the same as for local adic rings, we will only give a sketch here and refer the reader to [SP, Tag 0323] for more details.

Proposition 1.3.40. *Let $(A, \mathfrak{m})$ be a local quasi-adic ring. Then there exists a coefficient ring for A.*

Proof. If the characteristic of $A/\mathfrak{m}$ is 0, then A is necessarily equicharacteristic and we are done by Corollary 1.3.24. So assume that $\mathrm{char}(A/\mathfrak{m}) = p > 0$. We will make use of two facts:

1. For every field k with $\mathrm{char}(k) = p > 0$ there exists a complete discrete valuation ring Λ with uniformizer p and such that $\Lambda/p\Lambda \simeq A/\mathfrak{m}$ (see [SP, Tag 0328]).

2. For every such Λ the natural map $\mathbb{Z}/p^n\mathbb{Z} \to \Lambda/p^n\Lambda$ is formally smooth (see [SP, Tag 0329]).

Thus we take any such Λ and assume that we have constructed $\Lambda/p^n\Lambda \to A/\overline{\mathfrak{m}^{n+1}}$ inducing an isomorphism on residue fields. Since $\mathbb{Z}/p^n\mathbb{Z} \to \Lambda/p^n\Lambda$ is formally smooth there exists a diagonal arrow making the square

$$\begin{tikzcd}
\mathbb{Z}/p^n\mathbb{Z} \arrow[r] \arrow[d] & A/\overline{\mathfrak{m}^{n+1}} \arrow[d] \\
\Lambda/p^{n+1}\Lambda \arrow[r] \arrow[ur, dashed] & A/\overline{\mathfrak{m}^{n}}
\end{tikzcd}$$

commute. Taking the limit we obtain a map $\Lambda \to A$ which induces an isomorphism of residue fields. Its image is a coefficient ring for A. $\square$

Corollary 1.3.41. *For every local quasi-adic ring $(A, \mathfrak{m})$ there exists a surjection*

$$\Lambda \mathbin{\widehat{\otimes}_{\mathbb{Z}}} \mathbb{Z}[[x_i \mid i \in I]] \to A,$$

with Λ a coefficient ring.

Remark 1.3.42. Note that, if $|I| = \infty$, then $\Lambda \mathbin{\widehat{\otimes}_{\mathbb{Z}}} \mathbb{Z}[[x_i \mid i \in I]] \not\simeq \Lambda[[x_i \mid i \in I]]$. Indeed, this is for much the same reason as was mentioned in Remark 1.2.3: consider the polynomial ring $\mathbb{Z}_p[x_n \mid n \in \mathbb{N}]$ with coefficients in the ring of p-adic integers. Its completion $A := \mathbb{Z}_p \mathbin{\widehat{\otimes}_{\mathbb{Z}}} \mathbb{Z}[[x_n \mid n \in \mathbb{N}]]$ is a quasi-adic ring with coefficient ring $\mathbb{Z}_p$. Observe that

$$f = \sum_{n \geq 1} x_n p^n \in A$$

is not an element of $\mathbb{Z}_p[[x_i \mid i \in I]]$.

Proof of Corollary 1.3.41. This is done as in the proof of part (1) of Theorem 1.3.33. By Proposition 1.3.40, there exists a coefficient ring $(\Lambda, p\Lambda)$ for A. Let $a_i \in A$, $i \in I$, be elements whose images in $\mathfrak{m}/\overline{\mathfrak{m}^2}$ form a basis. Consider the morphism

$$\tau : \Lambda[x_i \mid i \in I] \to A,\ x_i \mapsto a_i,$$

where we consider the left hand side as endowed the preadic topology with respect to the ideal $\mathfrak{m}' := p\Lambda + (x_i \mid i \in I)$. As τ is continuous, it extends to a map $\widehat{\tau} : \Lambda \mathbin{\widehat{\otimes}_k} k[[x_i \mid i \in I]] \to A$. By Lemma 1.2.23, the induced map on associated gradeds is given by

$$\mathrm{gr}(\tau) : \mathrm{gr}(\Lambda)[x_i \mid i \in I] \to \mathrm{gr}(A),$$

and is surjective by Lemma 1.3.9. Thus so, by Proposition 1.2.13, τ is surjective. $\square$

1.4 Flatness of completion

It is a universally well-known fact that the completion of a finitely generated module over a Noetherian adic ring A is flat. Less well-known is that the completion of any (not necessarily finitely generated) *flat* module over A is flat again; a proof of this result can be found for example in [Yek18] or [SP, Tag 0AGW] . In this section we are considering a different situation: namely, when the ring A is no longer Noetherian. In this case, partial results using a weaker version of flatness were obtained in [Yek18, Theorem 0.4]. Our main result in this section shows that the completion of an infinite-variate polynomial ring over a field is flat. On the negative side, we give an example of a non-Noetherian local separated ring $(A, \mathfrak{m})$ whose $\mathfrak{m}$-adic completion $\widehat{A}$ is not flat, see Example 1.4.9. This demonstrates that flatness of the completion map is a rather subtle question in the non-Noetherian case.

Let A be a ring and $\mathfrak{m}$ an ideal in A. Given an A-module E we will consider the $\mathfrak{m}$-adic topology on E and we will denote by $\widehat{E}$ its $\mathfrak{m}$-adic completion. We are interested in conditions guaranteeing that the natural map $A \to \widehat{A}$ is flat.

Definition 1.4.1. Let E be an A-module and F a submodule of E. We say that $F \subseteq E$ has the *Artin–Rees property* with respect to $\mathfrak{m}$ if there exists a $c \in \mathbb{N}$ such that, for all $n > c$, we have

$$\mathfrak{m}^n E \cap F = \mathfrak{m}^{n-c}(\mathfrak{m}^c E \cap F).$$

The smallest such c is called the *Artin–Rees index* of $F \subseteq E$ with respect to $\mathfrak{m}$. We say that A has the *Artin–Rees property* with respect to $\mathfrak{m}$ if so does every finitely generated submodule of a finitely generated free A-module.

The Artin–Rees property for $F \subseteq E$ guarantees that the $\mathfrak{m}$-adic topology of F coincides with the topology induced by the $\mathfrak{m}$-adic topology of E. In this context it is natural to consider the Rees algebra $A^* = \bigoplus_{n \geq 0} \mathfrak{m}^n$ and the graded A^*-modules

$$E^* = \bigoplus_{n \geq 0} \mathfrak{m}^n E \qquad \text{and} \qquad F^* = \bigoplus_{n \geq 0} \mathfrak{m}^n E \cap F.$$

Lemma 1.4.2. *$F \subseteq E$ has the Artin–Rees property if and only if there exists a $c \in \mathbb{N}$ such that F^* is generated as a graded A^*-module by elements of degree $\leq c$. Moreover, the Artin–Rees index of $F \subseteq E$ is the smallest such c.*

Proof. This is immediate from the definitions. Compare with [Bou72, Chapter III, Section 3.1, Theorem 1] or [Mat89, Theorem 8.5] or [AM16, Lemma 10.8], but notice that no finite generation hypotheses are needed for the statement of the lemma. □

Remark 1.4.3. By the classical Artin–Rees lemma [Mat89, Theorem 8.5], any Noetherian ring A has the Artin–Rees property with respect to any ideal $\mathfrak{m} \subset A$. By contrast, there exist non-Noetherian rings, even finite dimensional, which do not have the Artin–Rees property. A zero-dimensional example is given by

$$A = k[x_i \mid i \in \mathbb{N}]/(x_1 - x_m^m \mid m \geq 2) + (x_n^{n+1} \mid n \geq 1),$$

with $\mathfrak{m} = (x_i \mid i \in \mathbb{N})$ and $F = (x_1) \subset E = A$. Clearly $x_1 \in \mathfrak{m}^n$ for all n, but there is no $f \in \mathfrak{m}$ such that $x_1 = x_1 f$.

In complete analogy with the Noetherian case, we prove that the Artin–Rees property implies flatness of the completion. We recall that a ring is *coherent* if every finitely generated ideal is finitely presented.

Proposition 1.4.4. *Let A be a coherent ring with the Artin–Rees property with respect to $\mathfrak{m} \subset A$, and let $\widehat{A}$ be its $\mathfrak{m}$-adic completion. Then $A \to \widehat{A}$ is flat. Moreover, if $\mathfrak{a} \subset A$ is a finitely generated ideal, then $\mathfrak{a}\widehat{A}$ is closed in $\widehat{A}$ (that is, $\mathfrak{a}\widehat{A} = \widehat{\mathfrak{a}}$).*

Proof. Let $\mathfrak{a}$ be a finitely generated ideal of A. Since A is coherent, there exists an exact sequence

$$A^p \longrightarrow A^q \xrightarrow{\varphi} \mathfrak{a} \longrightarrow 0.$$

Moreover, since the Artin–Rees property holds for $\ker\varphi \subset A^q$, the $\mathfrak{m}$-adic topology on $\ker\varphi$ agrees with the one induced by the inclusion $\ker\varphi \subset A^q$. From [Bou72, Chapter III, Section 2.12, Lemma 2] or [AM16, Lemma 10.3], the sequence remains exact after taking $\mathfrak{m}$-adic completions, and we have a commutative diagram

$$\begin{tikzcd}
A^p \otimes_A \widehat{A} \arrow[r] \arrow[d] & A^q \otimes_A \widehat{A} \arrow[r] \arrow[d] & \mathfrak{a} \otimes_A \widehat{A} \arrow[r] \arrow[d] & 0 \\
\widehat{A^p} \arrow[r] & \widehat{A^q} \arrow[r, "\widehat{\varphi}"] & \widehat{\mathfrak{a}} \arrow[r] & 0
\end{tikzcd}$$

with exact rows. Since taking completion commutes with finite direct sums, the map $\mathfrak{a} \otimes_A \widehat{A} \to \widehat{\mathfrak{a}}$ is an isomorphism. As the natural map $\widehat{\mathfrak{a}} \to \widehat{A}$ is an injection, flatness of $A \to \widehat{A}$ follows from [Mat89, Theorem 7.7]. The fact that $\mathfrak{a} \otimes_A \widehat{A} \to \widehat{\mathfrak{a}}$ is an isomorphism also shows that $\mathfrak{a}\widehat{A} = \widehat{\mathfrak{a}}$. ☐

The following theorem gives a first example of a non-Noetherian ring with the Artin–Rees property. We were not able to find a reference for this statement in the literature.

Theorem 1.4.5. *Let S be a Noetherian ring and $\mathfrak{n}$ any ideal of S. For any set I consider $P = S[x_i \mid i \in I]$ and $\mathfrak{m} = (x_i \mid i \in I) + \mathfrak{n}$. Then P has the Artin–Rees property with respect to $\mathfrak{m}$.*

Proof. Let E be a finitely generated free P-module and $F \subseteq E$ a finitely generated submodule. Assume that E is freely generated by $e_1, \dots, e_s$

Given any subset $J \subseteq I$, we write $P_J := S[x_i \mid i \in J]$, and for any ideal $\mathfrak{a} \subseteq P$ we denote $\mathfrak{a}_J := \mathfrak{a} \cap P_J$. We define $E_J := P_J \cdot e_1 \oplus \cdots \oplus P_J \cdot e_s$, and for any P-submodule $G \subseteq E$ we write $G_J := E_J \cap G$. Note that $P, \mathfrak{m}, \mathfrak{a}, E, G$ are the colimits of $P_J, \mathfrak{m}_J, \mathfrak{a}_J, E_J, G_J$ for $J \subseteq I$ finite. We have

$$G_J \cap G'_J = (G \cap G')_J,\ \mathfrak{a}_J G_J \subseteq (\mathfrak{a}G)_J,\ \mathfrak{a}_J E_J = (\mathfrak{a}E)_J,\ \text{and } (\mathfrak{m}_J)^n = (\mathfrak{m}^n)_J.$$

In particular, for all $n, d \in \mathbb{N}$ with $n > d$, we have

$$\mathfrak{m}_J^n E_J \cap F_J = (\mathfrak{m}^n E \cap F)_J \text{ and } \mathfrak{m}_J^{n-d}(\mathfrak{m}_J^d E_J \cap F_J) \subseteq (\mathfrak{m}^{n-d}(\mathfrak{m}^d E \cap F))_J.$$

Assume that F is generated by $f_1, \dots, f_r$. Then there exists a finite set $L \subset I$ such that $f_1, \dots, f_r \in F_L$, and for any J with $L \subseteq J \subseteq I$ we have $F_J = P_J \cdot f_1 + \cdots + P_J \cdot f_r = P_J \cdot F_L$.

Since P_L is Noetherian it has the Artin–Rees property with respect to $\mathfrak{m}_L$, and hence there exists a $c \in \mathbb{N}$ such that

$$\mathfrak{m}_L^n E_L \cap F_L = \mathfrak{m}_L^{n-c}(\mathfrak{m}_L^c E_L \cap F_L)$$

for all $n > c$. The smallest such c is the Artin–Rees index of $F_L \subseteq E_L$. Since for any finite set J with $L \subseteq J \subset I$ we have $F_J = P_J \cdot F_L$, we can apply Lemma 1.4.6 and we see that the Artin–Rees index of $F_J \subseteq E_J$ is again c. This implies that

$$(\mathfrak{m}^n E \cap F)_J \subseteq (\mathfrak{m}^{n-c}(\mathfrak{m}^c E \cap F))_J.$$

Taking the colimit for all finite $J \subset I$ we get that

$$\mathfrak{m}^n E \cap F \subseteq \mathfrak{m}^{n-c}(\mathfrak{m}^c E \cap F).$$

The reversed inclusion is immediate, and the theorem follows. □

Lemma 1.4.6. *Let A_0 be a Noetherian ring, $\mathfrak{m}_0 \subset A_0$ an ideal, E_0 a finitely generated A_0-module, and $F_0 \subseteq E_0$ a submodule. Let z be a new variable and consider the ring $A = A_0[z]$, the ideal $\mathfrak{m} = \mathfrak{m}_0 A + (z)$, the extension $E = A \otimes_{A_0} E_0 = E_0[z]$, and $F = A \otimes_{A_0} F_0 = F_0[z]$. Then the Artin–Rees index of $F \subseteq E$ with respect to $\mathfrak{m}$ equals the Artin–Rees index of $F_0 \subseteq E_0$ with respect to $\mathfrak{m}_0$.*

Proof. Let c_0 and c be the Artin–Rees indexes of $F_0 \subseteq E_0$ and $F \subseteq E$. As in Lemma 1.4.2, consider the Rees algebras

$$A_0^* = \bigoplus_{n \geq 0} \mathfrak{m}_0^n \quad \text{and} \quad A^* = \bigoplus_{n \geq 0} \mathfrak{m}^n,$$

and the graded modules

$$F_0^* = \bigoplus_{n \geq 0} \mathfrak{m}_0^n E_0 \cap F_0 \quad \text{and} \quad F^* = \bigoplus_{n \geq 0} \mathfrak{m}^n E \cap F.$$

Then F_0 is generated in degree $\leq c_0$ as a graded A_0-algebra (and not in any lower degree), and similarly for F.

Any element $f \in \mathfrak{m}^n E \cap F$ can be written as $f = \sum_{i=0}^n f_i z^{n-i}$ where $f_i \in \mathfrak{m}_0^i E_0 \cap F_0$. In particular, F^* is generated by F_0^* as an A^*-algebra, and therefore $c \leq c_0$. Conversely, if F^* is generated by homogeneous elements $f^{(1)}, \dots, f^{(r)}$ with $f^{(j)} = \sum_i f_i^{(j)} z^{n_j - i}$, then F_0^* is generated by $f_{n_1}^{(1)}, \dots, f_{n_r}^{(r)}$. We see that $c_0 \leq c$, and the result follows. □

Remark 1.4.7. If A and $\mathfrak{m}$ are as in Remark 1.4.3 then we have $A = \operatorname{colim}_m A_m$ where

$$A_m = k[x_1, \dots, x_m]/(x_1 - x_i^i, x_i^{i+1}, \ 1 < i \leq m).$$

It is easy to check that the Artin–Rees index of $(x_1) \subset A_m$ is m and A does not have the Artin–Rees property.

Corollary 1.4.8. *Let $S \to T$ be a map of Noetherian rings. As above, suppose that S has the discrete topology, and let T be equipped with the $\mathfrak{n}$-adic topology where $\mathfrak{n} \subset T$ is an ideal. Then the natural map*

$$T[x_i \mid i \in I] = T \otimes_S S[x_i \mid i \in I] \to T \mathbin{\widehat{\otimes}_S} S[x_i \mid i \in I]$$

is flat. In particular:

1. *for any index set I the completion map $S[x_i \mid i \in I] \to S[[x_i \mid i \in I]]$ is flat, and*

2. *for every finite subset $J \subset I$ the inclusion $S[[x_j \mid j \in J]] \to S[[x_i \mid i \in I]]$ is flat.*

Proof. By Proposition 1.2.21 a basis for the topology on $T[x_i \mid i \in I]$ is given by the filtration

$$\sum_{n+m=d} \mathfrak{n}^m[x_i \mid i \in I] + (x_i \mid i \in I)^n, \quad d \in \mathbb{N},$$

which is easily seen to be equivalent to the $\mathfrak{m}$-adic one, where $\mathfrak{m} := (x_i \mid i \in I) + \mathfrak{n}$. As $T[x_i \mid i \in I]$ is coherent, the first assertion follows from Theorem 1.4.5 and Proposition 1.4.4. Regarding the last two assertions, (1) follows by observing that $S \mathbin{\widehat{\otimes}}_S S[x_i \mid i \in I] = S[[x_i \mid i \in I]]$, and (2) by taking $T = S[[x_j \mid j \in J]]$ with the $(x_j \mid j \in J)$-adic topology and observing that the given inclusion factors as

$$S[[x_j \mid j \in J]] \to S[[x_j \mid j \in J]][x_i \mid i \in I \setminus J] \to S[[x_i \mid i \in I]]$$

and so is flat. □

The following example shows that extending Corollary 1.4.8 beyond polynomial rings is rather delicate.

Example 1.4.9. For quotients A of $k[x_i \mid i \in I]$ the completion map $A \to \widehat{A}$ need not be flat, even if the topology of A is separated. Consider the ideal

$$\mathfrak{a} = (yx_1, yx_n^n - zx_{n-1}^{n-1} \mid n > 1)$$

in $P = k[x_n, y, z \mid n \in \mathbb{N}_{>0}]$ and the quotient $A = P/\mathfrak{a}$. Let $\mathfrak{m} = (x_n, y, z \mid n \in \mathbb{N}_{>0}) \subset A$. As $\mathfrak{m}$ is weighted homogeneous with respect to the positive weights $w(x_n) = w(y) = 1$, $w(z) = 2$, it follows that the $\mathfrak{m}$-adic topology on A is separated. Consider the element $y - z$, which is annihilated by the series $f = \sum_{n \geq 1} x_n^n$. If $\widehat{A}$ were flat over A, there would exist polynomials $a_1, \ldots, a_r \in A$ annihilating $y - z$ such that f can be written as $f = \sum_{j=1}^r a_j b_j$ where $b_j \in \widehat{A}$.

Considering this equation modulo (y, z), we have written f as a linear combination of polynomials in $k[x_n \mid n \in \mathbb{N}_{>0}]$, which is clearly impossible.

We close this section with the following analogue to Proposition 1.2.16 for polynomial rings.

Proposition 1.4.10. *Let $P = S[x_i \mid i \in I]$ and $\mathfrak{m} = (x_i \mid i \in I)$, where S is a ring and I a set. Let $\mathfrak{a} \subset P_\mathfrak{m}$ be an ideal such that $\operatorname{in}(\mathfrak{a}) \subset P$ is finitely generated. Then $\mathfrak{a}$ is finitely generated.*

Proof. Let $f_1, \ldots, f_r \in \mathfrak{a}$ be such that $\operatorname{in}(f_1), \ldots, \operatorname{in}(f_r)$ generate $\operatorname{in}(\mathfrak{a})$. Since $\operatorname{in}(\mathfrak{a}) = \operatorname{in}(\widehat{\mathfrak{a}})$, we can apply Proposition 1.2.16 to see that $\widehat{\mathfrak{a}} = (f_1, \ldots, f_r)\widehat{P}$. By Corollary 1.4.8 the map $P_\mathfrak{m} \to \widehat{P}$ is faithfully flat and thus $\mathfrak{a} \subset \widehat{\mathfrak{a}} \cap P_\mathfrak{m} = (f_1, \ldots, f_r)P_\mathfrak{m}$. The other inclusion is trivial, so $\mathfrak{a} = (f_1, \ldots, f_r)$. □

1.5 Standard bases and the division theorem

Standard bases of ideals of (finite-variate) formal power series are a classical tool both in computational algebra as well as in singularity theory. Even though the principal ideas remain the same, some care has to be taken when extending results to the infinite-variate case. Our goal in this section is to give a proof of both the Grauert–Hironaka division theorem (see [Gra72; Hir77]) as well as a version of the Buchberger criterion for infinite-variate power series rings. We also refer the reader to [Bec90] whose proof we follow closely here. As we do not make explicit use of the results of this section anywhere else in this monograph, the reader should consider this section purely expository in nature.

Throughout this section we will always assume k to be a field - not necessarily of characteristic 0 - and I to be an index set of arbitrary size. In order to be more in line with other works on standard bases we will change our notation in this section. That is, we write $\mathcal{O} := k[x_i \mid i \in I]$ for the polynomial ring in indeterminates x_i, $i \in I$, as well as $\mathfrak{m} := (x_i \mid i \in I)$ for the maximal ideal of $\mathcal{O}$ generated by all the x_i. Moreover, write $\widehat{\mathcal{O}} := k[[x_i \mid i \in I]]$ for the corresponding power series ring over k. For any subset $J \subset I$ we will write $\mathcal{O}(J) := k[x_j \mid j \in J]$ and $\widehat{\mathcal{O}}(J) := k[[x_j \mid j \in J]]$. By $M(I)$ we denote the set of monomials in the indeterminates x_i, $i \in I$, that is, $M(I) := \{x^\alpha \mid \alpha \in \mathbb{N}^{(I)}\}$.

1.5.1 Weighted degrees and filtrations

In this section we will briefly introduce weighted degrees and their associated filtrations on the rings $\mathcal{O}$ and $\widehat{\mathcal{O}}$. For I finite, these notions are of course well-known. In the general case however we need to introduce some conditions to ensure compatibility with the inverse limit structure of $\widehat{\mathcal{O}}$.

Let $\omega : I \to \mathbb{R}_{>0}$ be a function and write $\omega_i := \omega(i)$. We say that ω is a *weight system* on I (or $M(I)$) if it is bounded by some $a, b > 0$, i.e for all $i \in I$

$$a < \omega_i < b.$$

For a monomial $x^\alpha \in M(I)$ we define its ω-degree as $\deg_\omega(x^\alpha) := \omega \cdot \alpha = \sum_i \omega_i \alpha_i$. We define the ω-order of a polynomial $f = \sum_{\alpha \in \mathbb{N}^{(I)}} f_\alpha x^\alpha \in \mathcal{O}$ to be

$$\operatorname{ord}_\omega(f) := \min\{\omega \cdot \alpha,\ f_\alpha \neq 0\}.$$

The ω-order in turn gives us a descending $\mathbb{N}$-filtration of $\mathcal{O}$ by ideals

$$\mathcal{O}_{\omega,d} := \{f \in \mathcal{O} : \operatorname{ord}_\omega(f) \geq d\},$$

Now we have the following result:

Lemma 1.5.1. *If ω is a weight system, then the filtration $\{\mathcal{O}_{\omega,d}\}_{d \in \mathbb{N}}$ is equivalent to the $\mathfrak{m}$-adic filtration of $\mathcal{O}$.*

Proof. Let $a, b > 0$ be such that $a < \omega_i < b$ for all $i \in I$. For $e > 0$ choose d such that $ad > e$, then $\mathcal{O}_{\omega,d} \subset \mathfrak{m}^e$. Conversely, for $d > 0$ choose e such that $db < e$. Thus $\mathfrak{m}^e \subset \mathcal{O}_{\omega,d}$. □

Corollary 1.5.2. *The completions of $\mathcal{O}$ with respect to the $\mathfrak{m}$-adic filtration and the filtration $\{\mathcal{O}_{\omega,d}\}$ agree. In particular, there exist* truncation maps

$$\pi_\omega^d : \widehat{\mathcal{O}} \to \mathcal{O}/\mathcal{O}_{\omega,d}$$

such that a power series $f = \sum_{\alpha\in\mathbb{N}^{(I)}} f_\alpha x^\alpha \in \widehat{\mathcal{O}}$ *gets mapped to*

$$\pi_\omega^d(f) = \sum_{\omega\cdot\alpha<d} f_\alpha x^\alpha.$$

Example 1.5.3. Let $I = \mathbb{N}$, then one particularly useful choice of weight system is the following: let $p_i \in \mathbb{Z}_{>0}$ be pairwise coprime positive integers and $c_i := \lceil \log(p_i) \rceil$. By construction $\omega_i := \log(p_i)/c_i$ gives a weight system, with the additional property that for any given $e \in \mathbb{R}_{>0}$ there is at most one monomial in $M(I)$ of degree e. In fact, such a choice of weight system already gives a monomial order, as seen later in Example 1.5.4.

1.5.2 Standard bases in power series rings

The goal of this section is to prove Buchberger's criterion for standard bases of infinite-variate power series by means of the Grauert–Hironaka division theorem. The main technical obstacle of working with standard bases in power series rings, when compared to Gröbner bases in polynomial rings, is to ensure compatibility with the inverse limit structure. In particular, we will need to consider special types of monomial orders. Recall that a *monomial order* on $M(I)$ is a total ordering $<$ of $M(I)$ satisfying:

$$x^\alpha < x^\beta \Rightarrow x^\gamma x^\alpha < x^\gamma x^\beta,\ \forall x^\gamma \in M(I).$$

We will identify such any monomial order $<$ with the induced (additive) order on $\mathbb{N}^{(I)}$ and use them interchangibly.

Let ω be a weight system on I. Then a monomial order $<$ is called *admissible* (with respect to ω) if, for every $\alpha, \beta \in \mathbb{N}^{(I)}$ we have

$$\deg_\omega(x^\alpha) \geq \deg_\omega(x^\beta) \Rightarrow x^\alpha \leq x^\beta.$$

So monomials of smaller degrees will be larger with respect to $<$. Such $<$ are often called a *local monomial orders* in the literature (see [Greuel Pfister]). Admissible orders will not be well orderings, in contrast to the monomial orders usually used in Gröbner basis applications.

Example 1.5.4. Let $I = \mathbb{N}$ be countable and consider the weight system of Example 1.5.3. Define a monomial order on $N^{(I)}$ via

$$\alpha <_\omega \beta :\Leftrightarrow \omega\cdot\alpha > \omega\cdot\beta.$$

Then $<_\omega$ is an admissible monomial order with respect to the filtration induced by ω.

Remark 1.5.5. If $|I| = \infty$ then there are admissible monomial orders $<$ such that there exist infinite chains of monomials

$$\ldots < x^{\alpha_i} < x^{\alpha_{i-1}} < \ldots < x^{\alpha_0},$$

all of which have the same ω-degree. For example, choose $I = \mathbb{N}$ and $<$ the degree lexicographic ordering where

$$x_1 > x_2 > \ldots$$

In fact, this phenomenon is the main technical obstruction when adapting the classical proof of the Division theorem to the infinite-variate case (see Theorem 1.5.6 and in particular Lemma 1.5.7). One could bypass this by a suitable choice of admissible monomial order $<$; however, we do want to allow monomial orders such as the degree lexicographic ordering which are compatible with the standard filtration on $\mathcal{O}$.

For the rest of this section let ω be a weight system and fix a monomial order $<$ admissible with respect to ω. For any series $f \in \widehat{\mathcal{O}}$ there exists a $\Lambda \subset \mathbb{N}^{(I)}$ such that f may be written as

$$f = \sum_{\alpha \in \Lambda} f_\alpha x^\alpha, f_\alpha \neq 0.$$

We call Λ the *support of* f and write $\operatorname{supp}(f) = \Lambda$. Now let β be the $<$-largest element of f, which always exists by Corollary 1.5.2. We define:

- $\mathrm{lt}_<(f) := f_\beta x^\beta$, the leading term of f.
- $\mathrm{lm}_<(f) := x^\beta$, the leading monomial of f.
- $\mathrm{lc}_<(f) := f_\beta$, the leading coefficient of f.
- $\mathrm{lexp}_<(f) := \beta$, the leading exponent of f.
- $\mathrm{tail}_<(f) := f - \mathrm{lt}(f)$, the tail of f.

We will omit the $<$ in the index whenever there is no ambiguity.

Throughout the following, we fix an admissible monomial order $<$. For any given $f_1, \ldots, f_m \in \widehat{\mathcal{O}}$, let $\alpha_i := \mathrm{lexp}(f_i)$ and define Δ to be the submonoid of $\mathbb{N}^{(I)}$ generated by $\alpha_1, \ldots, \alpha_m$, written $\Delta = \langle \alpha_1, \ldots, \alpha_m \rangle$. Denote the complement of Δ in $\mathbb{N}^{(I)}$ as Δ_c. We define submonoids $\Delta_1, \ldots, \Delta_m \subset \mathbb{N}^{(I)}$ recursively as

$$\Delta_1 := \langle \alpha_1 \rangle,\ \Delta_i := \langle \alpha_i \rangle \setminus \langle \alpha_1, \ldots, \alpha_{i-1} \rangle$$

and finally

$$\Gamma_i := \begin{cases} \Delta_i - \alpha_i,\ \Delta_i \neq \emptyset, \\ 0,\ \text{else} \end{cases}$$

For any subset $\Lambda \subset \mathbb{N}^{(I)}$ we write

$$\widehat{\mathcal{O}}_\Lambda := \{ f \in \widehat{\mathcal{O}} : \operatorname{supp}(f) \subset \Lambda \}.$$

For example, no term of $f \in \widehat{\mathcal{O}}_{\Delta_c}$ is divisible by x^{α_i} for some $i = 1, \ldots, m$, while $\widehat{\mathcal{O}}_{\{0\}} = k$.

The above definitions are classical in the case where $|I| < \infty$ and are used to prove a variant of the Grauert–Hironaka division theorem. For any infinite I the same proof works with minor modifications, the latter of which could be omitted altogether by considering only particular monomial orders (see Remark 1.5.5).

Theorem 1.5.6 (Grauert–Hironaka division). *Let $f_1, \ldots, f_m \in \widehat{\mathcal{O}}$ and $<$ be any admissible monomial order with respect to some weight system ω. Consider the operator*

$$\Phi : \widehat{\mathcal{O}}_{\Gamma_1} \times \ldots \times \widehat{\mathcal{O}}_{\Gamma_m} \times \widehat{\mathcal{O}}_{\Delta_c} \to \widehat{\mathcal{O}},$$

$$(q_1, \ldots, q_m, r) \mapsto \sum_{i=1}^m q_i f_i + r,$$

where $\widehat{\mathcal{O}}_{\Gamma_i}$ and $\widehat{\mathcal{O}}_{\Delta_c}$ are defined as before. Then Φ is invertible.

Proof. Consider the map $\Psi : \widehat{\mathcal{O}}_{\Gamma_1} \times \ldots \times \widehat{\mathcal{O}}_{\Gamma_m} \times \widehat{\mathcal{O}}_{\Delta_c} \to \widehat{\mathcal{O}}$ given by

$$\Psi(q_1, \ldots, q_m, r) := \sum_{i=1}^{m} q_i x^{\alpha_i} + r,$$

which is clearly an isomorphism. We will show that $\Phi \circ \Psi^{-1}$ is invertible by proving that $\sum_n \Theta^n$ converges, where $\Theta := \mathrm{id} - \Phi \circ \Psi^{-1}$. For $f \in \widehat{\mathcal{O}}$, write $f = \sum q_i x^{\alpha_i} + r$ with $q_i \in \widehat{\mathcal{O}}_{\Gamma_i}$ and $r \in \widehat{\mathcal{O}}_{\Delta_c}$. Then

$$\Theta(f) = \sum_{i=1}^{m} q_i (x^{\alpha_i} - f_i),$$

and thus $\mathrm{lm}(\Theta(f)) < \mathrm{lm}(f)$. Clearly $\mathrm{ord}_\omega(\Theta(f)) \geq \mathrm{ord}_\omega(f)$. It is sufficient to prove that, for some $n > 0$, we have $\mathrm{ord}_\omega(\Theta^n(f)) > \mathrm{ord}_\omega(f)$. To prove this assertion, assume the contrary. As before, we write $\pi^d = \pi^d_\omega : \widehat{\mathcal{O}} \to \mathcal{O}/\mathcal{O}_{\omega,d}$ for the truncation map. We will make use of the following lemma.

Lemma 1.5.7. *For all $d > 0$ there exists a finite subset $J \subset I$, depending only on $f_1, \ldots, f_m$, such that: if $\pi^d(f) \in \widehat{\mathcal{O}}(L)$ for some $L \subset I$, then $\pi^d(\Theta(f)) \in \widehat{\mathcal{O}}(L \cup J)$.*

Proof. Choose $J \subset I$ finite such that $\pi^d(f_1), \ldots, \pi^d(f_m) \in \widehat{\mathcal{O}}(J)$. Consider $\Psi^{-1}(f) = \sum_{i=1}^m q_i x^{\alpha_i} + r$ and write $q_i = q_i' + q_i''$ with $q_i' \in \widehat{\mathcal{O}}(L)$ and $q_i' x^{\alpha_i} = \pi^d(q_i x^{\alpha_i})$. Then $\pi^d(q_i' f_i) = \pi^d(q_i f_i)$. Thus

$$\pi^d(\Theta(f)) = \sum_{i=1}^{m} \underbrace{\pi^d(q_i x^{\alpha_i})}_{\in \widehat{\mathcal{O}}(L)} - \underbrace{\pi^d(q_i f_i)}_{\in \widehat{\mathcal{O}}(L \cup J)}.$$

□

Choose a finite $L \supset J$ such that $\pi^d(f) \in \widehat{\mathcal{O}}(L)$, then by the above we have that $\pi^d(\Theta^n(f)) \in \widehat{\mathcal{O}}(L)$ for every $n > 0$. This implies that there exists an infinite chain of monomials in $M(L)$

$$\ldots < \mathrm{lm}(\Theta^{n+1}(f)) < \mathrm{lm}(\Theta^n(f)) < \ldots < \mathrm{lm}(f),$$

all of the same ω-degree. Since L is finite, this is impossible. □

Remark 1.5.8. In fact, both Φ and its inverse from the theorem are *textile* in the sense of [BH10; HW], i.e. the coefficients of $\Phi(f)$ are polynomials in the coefficients of f.

Definition 1.5.9. For any ideal $\mathfrak{a}$ of $\widehat{\mathcal{O}}$ the *initial ideal* of $\mathfrak{a}$ (with respect to the monomial order $<$) is defined as

$$\mathrm{in}_<(\mathfrak{a}) := (\mathrm{lt}_<(f),\ f \in \mathfrak{a}) \subset \widehat{\mathcal{O}}.$$

A finite set $\mathcal{F} = \{f_1, \ldots, f_m\}$ of elements of $\mathfrak{a}$ is called a *standard basis* if the corresponding set of leading terms $\{\mathrm{lt}_<(f_1), \ldots, \mathrm{lt}_<(f_m)\}$ generates $\mathrm{in}_<(\mathfrak{a})$.

As before, we will omit the $<$ in the index whenever there is no ambiguity.

Remark 1.5.10. In Theorem 1.5.11 we will prove that any standard basis of an ideal of $\widehat{\mathcal{O}}$ already generates it. As we require standard bases to be finite, that implies that there does not exist a standard basis for non-finitely generated ideals, for example the maximal ideal of $\widehat{\mathcal{O}}$ when $|I| = \infty$.

On the other hand, to our knowledge, the question of whether a finitely generated ideal of $\widehat{\mathcal{O}}$ always has a standard basis is still open. As mentioned in Remark 1.2.1, for $|I| = \infty$ the ring $\widehat{\mathcal{O}}$ is not the colimit over its finite-variate power series subrings, hence we cannot immediately deduce the existence of a standard basis for general finitely generated ideals $\mathcal{O}$ from the finite-variate case. However, if we limit ourselves to ideals $\mathfrak{a}$ which are generated by $f_1, \ldots, f_r \in \widehat{\mathcal{O}}(J)$ for $J \subset I$ finite, then we will see in Corollary 1.5.13 that any standard basis for the ideal $\mathfrak{a} \cap \widehat{\mathcal{O}}(J)$ will give one for $\mathfrak{a}$ itself. We will call such ideals $\mathfrak{a}$ *of finite definition* and they will be the main subject of Section 1.6.

The division theorem is the main ingredient in proving the Buchberger criterion for standard bases in power series ring. Let us briefly recall the definition of *s-pair* in our setting: for $f, g \in \widehat{\mathcal{O}}$ we write $x^\alpha := \operatorname{lm}(f)$, $x^\beta := \operatorname{lm}(g)$ and $x^\gamma := \operatorname{lcm}(x^\alpha, x^\beta)$. Then the s-pair of f, g is defined to be the series

$$S(f,g) := \operatorname{lc}(g) x^{\gamma-\alpha} f - \operatorname{lc}(f) x^{\gamma-\beta} g.$$

Theorem 1.5.11 (Buchberger criterion). *Let $\mathfrak{a} \subset \widehat{\mathcal{O}}$ be an ideal and $\mathcal{F} = \{f_1, \ldots, f_m\} \subset \mathfrak{a}$. Then the following are equivalent*

1. *$\mathcal{F}$ is a standard basis of $\mathfrak{a}$.*
2. *$\mathcal{F}$ generates $\mathfrak{a}$ and every $g \in \widehat{\mathcal{O}}$ has a unique normal form with respect to $\mathcal{F}$, i.e. there exist $q_1, \ldots, q_m \in \widehat{\mathcal{O}}$ and a unique $r \in \widehat{\mathcal{O}}$ such that*
$$g = \sum_{i=1}^{m} q_i f_i + r$$
and no monomial of r is divisible by any $\operatorname{lt}(f_i)$.
3. *Every $f \in \mathfrak{a}$ has standard representation with respect to $\mathcal{F}$, i.e. there exist $q_1, \ldots, q_m \in \widehat{\mathcal{O}}$ with $f = \sum_{i=1}^{m} q_i f_i$ and $\operatorname{lt}(f) \geq \operatorname{lt}(q_i)\operatorname{lt}(f_i)$.*
4. *Every s-pair $S(f_i, f_j)$ has standard representation with respect to $\mathcal{F}$ for all i, j.*

Proof. The proof of the equivalence of (1)-(3) is exactly the same as in the classical case and we shall reproduce it here. First, let us see that (1) implies (3). If $f \in \mathfrak{a}$ then Theorem 1.5.6 implies that

$$f = \sum_{i=1}^{m} q_i f_i + r,$$

with no monomial of r being divisible by x^{α_i}. But $r \in \mathfrak{a}$ and thus $\operatorname{lt}(r) \in \operatorname{in}(\mathfrak{a}) = (x^{\alpha_1}, \ldots, x^{\alpha_m})$, which implies $r = 0$.

Now assume (3) and let us prove (2). Suppose g may be written as $g = \sum_i q_i f_i + r = \sum_i \widetilde{q}_i f_i + \widetilde{r}$. Then

$$r - \widetilde{r} = \sum_{i=1}^{m} (q_i - \widetilde{q}_i) f_i$$

and we get $\operatorname{lt}(r - \widetilde{r}) = 0$ and thus $r - \widetilde{r} = 0$. Finally, to see that (2) implies (1) observe that 0 is the unique normal form for any $f \in \mathfrak{a}$ and hence $\operatorname{lt}(f)$ is divisible by some x^{α_i}.

(3) implies (4) is obvious, so we are left to show that (4) implies that $\mathcal{F}$ is a standard basis. We follow the argument given by [Bec90] and first prove that we may refine any presentation that is not standard by means of the s-pair.

Lemma 1.5.12. *Let $\mathcal{F} = \{f_1, \ldots, f_m\}$ be a subset of $\widehat{\mathcal{O}}$ such that, for all i,j, the s-pair $S(f_i, f_j)$ has standard representation with respect to $\mathcal{F}$. For every $d > 0$ there exists a finite subset $J \subset I$ with the following property: for every $f \in \widehat{\mathcal{O}}$ and presentation $f = \sum_i q_i f_i$ which is not standard, i.e.,*

$$\operatorname{lm}(f) < \min_i \operatorname{lm}(q_i) \operatorname{lm}(f_i)$$

and such that $\pi^d(f), \pi^d(q_i) \in \widehat{\mathcal{O}}(L)$ for some $L \subset I$, there exist $q'_1, \ldots, q'_m \in \widehat{\mathcal{O}}$ with $f = \sum_i q'_i f_i$, $\pi^d(q'_i) \in \widehat{\mathcal{O}}(L \cup J)$ and

$$\min_i \operatorname{lm}(q'_i) \operatorname{lm}(f_i) < \min_i \operatorname{lm}(q_i) \operatorname{lm}(f_i).$$

Proof. For all i, j fix a standard representation of $S(f_i, f_j)$, i.e. choose $p_l^{(i,j)} \in \widehat{\mathcal{O}}$ such that

$$S(f_i, f_j) = \sum_{l=1}^{m} p_l^{(i,j)} f_l.$$

Choose J such that $\pi^d(f_i), \pi^d(p_l^{(i,j)}) \in \widehat{\mathcal{O}}(J)$ for all i, j, l.

Let $f = \sum_{i=1}^{m} q_i f_i$ be a presentation which is not standard; we will prove the theorem by induction on the number k of $j \in \{1, \ldots, m\}$ such that

$$\operatorname{lm}(q_j) \operatorname{lm}(f_j) = \min_i \operatorname{lm}(q_i) \operatorname{lm}(f_i)$$

Since, by our assumptions, we have that $k \geq 2$, let us first prove the case $k = 2$. Then, without loss of generality, we may assume that

$$\operatorname{lm}(q_1) \operatorname{lm}(f_1) = \operatorname{lm}(q_2) \operatorname{lm}(f_2) = \min_i \operatorname{lm}(q_i) \operatorname{lm}(f_i).$$

Thus $\operatorname{lt}(q_1) \operatorname{lt}(f_1) = -\operatorname{lt}(q_2) \operatorname{lt}(f_2)$ and there exist $c \in k$ and $\alpha \in \mathbb{N}^{(I)}$ such that

$$\operatorname{lt}(q_1) f_1 + \operatorname{lt}(q_2) f_2 = c x^\alpha S(f_1, f_2) = c x^\alpha \sum_{l=1}^{m} p_l^{(1,2)} f_l.$$

Set

$$q'_i := q_i - \operatorname{lt}(q_i) + c x^\alpha p_i^{(1,2)},\ i = 1, 2,$$

and

$$q'_i := q_i + c x^\alpha p_i^{(1,2)},\ i > 2.$$

Clearly $\pi^d(q_i') \in \mathcal{O}(L \cup J)$ for all i. Furthermore, it is easy to check that $f = \sum_{i=1}^m q_i' f_i$. Finally, since

$$\operatorname{lm}(x^\alpha p_i^{(1,2)}) < \operatorname{lm}(q_1)\operatorname{lm}(f_1) = \operatorname{lm}(q_2)\operatorname{lm}(f_2),$$

we see that

$$\operatorname{lm}(q_i')\operatorname{lm}(f_i) \leq \operatorname{lm}(q_i)\operatorname{lm}(f_i),$$

with the inequality being strict for $i = 1, 2$.

Now assume the statement has been proven for $k-1$ and, again, assume without loss of generality that

$$\operatorname{lm}(q_1)\operatorname{lm}(f_1) = \operatorname{lm}(q_2)\operatorname{lm}(f_2) = \min_i \operatorname{lm}(q_i)\operatorname{lm}(f_i).$$

Then we write

$$f = q_1 f_1 - \frac{\operatorname{lc}(q_1 f_1)}{\operatorname{lc}(q_2 f_2)} q_2 f_2 + \left(\frac{\operatorname{lc}(q_1 f_1)}{\operatorname{lc}(q_2 f_2)} + 1\right) q_2 f_2 + \sum_{i=3}^m q_i f_i.$$

Repeating the above argument for the first two summands yields a presentation

$$f = r_1 f_1 + r_2 f_2 + \sum_{i=2}^m q_i' f_i$$

satisfying $r_i, q_i' \in \widehat{\mathcal{O}}(L \cup J)$ and

$$\begin{aligned} \operatorname{lm}(r_j)\operatorname{lm}(f_j) &< \operatorname{lm}(q_j)\operatorname{lm}(f_j), j = 1, 2 \\ \operatorname{lm}(q_i')\operatorname{lm}(f_i) &\leq \operatorname{lm}(q_i)\operatorname{lm}(f_i), i = 2, \ldots, m. \end{aligned}$$

Thus there exist at most $k-1$ indices i with

$$\operatorname{lm}(q_i')\operatorname{lm}(f_i) = \min_i \operatorname{lm}(q_i)\operatorname{lm}(f_i)$$

and we can apply induction to finish the proof. □

To prove the missing implication of Theorem 1.5.11, assume that there exists an $f \in \mathfrak{a}$ which does not have standard representation. Let $d \geq \operatorname{ord}_\omega(f)$ and J be the finite subset of I in the statement of Lemma 1.5.12. For $L \supset J$ finite we choose a presentation $f = \sum_{i=1}^m q_i f_i$ with $\pi^d(q_i) \in \widehat{\mathcal{O}}(L)$ and such that $\min_i \operatorname{lm}(q_i)\operatorname{lm}(f_i)$ is $<$-minimal among all such presentations. Note that for some L this minimum exists since $\operatorname{lm}(q_i)\operatorname{lm}(f_i) \in M(L)$ and there exist only finitely many monomials in $M(L)$ smaller than $\operatorname{lm}(f)$. Now applying Lemma 1.5.12 gives a contradiction to the minimality. □

Corollary 1.5.13. *Let $J \subset I$ be a finite subset and $\mathfrak{a} \subset \widehat{\mathcal{O}}(J)$ be an ideal. Then any standard basis for $\mathfrak{a}$ gives a standard basis for the extension $\mathfrak{a}^e := \mathfrak{a}\widehat{\mathcal{O}}$.*

Proof. Follows immediately from (4) of Theorem 1.5.11. □

Corollary 1.5.14. *Let ω be a weight system and $<$ an ω-admissible monomial ordering. Let $\mathfrak{a}$ be an ideal of $\widehat{\mathcal{O}}$. Then any standard basis of $\mathfrak{a}$ gives a set of generators for $\operatorname{in}_\omega(\mathfrak{a})$.*

Proof. Follows immediately from (3) of Theorem 1.5.11. □

1.6 Ideals of finite definition

In this section, we fix a field k and a set I, and consider the polynomial ring $P = k[x_i \mid i \in I]$ and the power series ring $\widehat{P} = k[[x_i \mid i \in I]]$. An important class of ideals in $\widehat{P}$ are those generated by finitely many power series involving only finitely many variables. This section is devoted to studying the properties of such ideals; in particular, we will prove that their height in $\widehat{P}$ is always finite, a fact we will make use of in Section 2.5.2.

For any subset $J \subset I$, we write $P_J = k[x_i \mid i \in J]$ and $\widehat{P}_J = k[[x_i \mid i \in J]]$, and for any ideal $\mathfrak{a} \subset \widehat{P}$ we denote $\mathfrak{a}_J := \mathfrak{a} \cap \widehat{P}_J$.

Definition 1.6.1. Let $\mathfrak{a} \subset \widehat{P}$ be an ideal.

1. We say that $\mathfrak{a}$ is *of finite definition* with respect to the indeterminates x_i if there exists a finite subset $J \subset I$ such that $\mathfrak{a} = \mathfrak{a}_J \widehat{P}$.

2. Similarly, $\mathfrak{a}$ is *of finite polynomial definition* with respect to the indeterminates x_i if it is generated by finitely many polynomials, i.e., elements in P.

3. We say that $\mathfrak{a}$ is *of finite (polynomial) definition* if there exist a isomorphism $\widehat{P} \simeq k[[x'_i \mid i \in I]]$ of topological k-algebras such that $\mathfrak{a}$ is of finite (polynomial) definition with respect to the formal coordinates x'_i.

Definition 1.6.2. Let $(A, \mathfrak{m}, k)$ be an equicharacteristic local quasi-preadic ring.

1. A *weak DGK decomposition* for A is an isomorphism $\widehat{A} \simeq k[[x_i \mid i \in I]]/\mathfrak{a}$ where $\mathfrak{a}$ is an ideal of finite definition.

2. A *DGK decomposition* for A is an isomorphism $\widehat{A} \simeq k[[x_i \mid i \in I]]/\mathfrak{a}$ with $\mathfrak{a}$ of finite polynomial definition.

3. We say that a (weak) DGK decomposition $\widehat{A} \simeq k[[x_i \mid i \in I]]/\mathfrak{a}$ is *efficient* if the quotient map $k[[x_i \mid i \in I]] \to \widehat{A}$ is an efficient formal embedding.

Remark 1.6.3. If A has a DGK decomposition, then we have an isomorphism $\widehat{A} \simeq \widehat{B} \widehat{\otimes}_k \widehat{P}$ where $\widehat{P}$ is a power series ring and $(B, \mathfrak{n}, k)$ is a local k-algebra which is essentially of finite type. Geometrically, this means that $\mathrm{Spf}(\widehat{A}) \cong \widehat{Z}_z \widehat{\times} \Delta^I$ where $\Delta^I = \mathrm{Spf}(k[[x_i \mid i \in I]])$ and $\widehat{Z}_z$ is the formal neighborhood of a scheme Z of finite type over k at a point $z \in Z(k)$. If A has a weak DGK decomposition, then $\widehat{A} \simeq \mathcal{B} \widehat{\otimes}_k \widehat{P}$ where $\mathcal{B}$ is a Noetherian complete local ring with residue field k.

Example 1.6.4. The existence of a weak DGK decomposition for a ring A does not imply the existence of a DGK decomposition for A. This can be seen by considering the following example given by Whitney. Let $f(t)$ be a transcendental power series with complex coefficients and with $f(0) = 0$, and consider the equation

$$g = xy(y - x)(y - (3 + t)x)(y - (4 + f(t))x).$$

It is proven in [Whi65, Example 14.1] that $\mathcal{B} = \mathbb{C}[[x, y, t]]/(g)$ is not isomorphic to the completion of a local ring of a $\mathbb{C}$-scheme of finite type. In particular, any local ring A for which $\widehat{A} \simeq \mathcal{B}$ (for example, $\mathcal{B}$ itself) admits a weak DGK decomposition but not a DGK decomposition.

We now give another example of a local ring A such that $\widehat{A} \simeq \mathcal{B}$. This example has the advantage of being explicitly presented as the localization of a quotient of a polynomial ring in countably many variables. Write $f(t) = \sum_{i\geq 1} a_i t^i \in \mathbb{C}[[t]]$. Consider the polynomial ring $P = \mathbb{C}[x, y, t, z_n \mid n \geq 0]$ and the ideal

$$\mathfrak{a} = (h, z_{n-1} - z_n t - a_n t \mid n \geq 1)$$

where

$$h = xy(y - x)(y - (3 + t)x)(y - (4 + z_0)x).$$

Let A be the localization of $P/\mathfrak{a}$ at the ideal $(x, y, t, z_n \mid n \geq 0)$. Then, in $\widehat{A}$, we have for each $m \geq 1$

$$z_0 - f(t) = z_m t^m - \sum_{i\geq m+1} a_i t^i \in \widehat{\mathfrak{m}}^m,$$

and for each $m \geq n + 1$

$$z_n - \sum_{i\geq n+1} a_i t^{i-n} = z_m t^{m-n} - \sum_{i\geq m+1} a_i t^{i-n} \in \widehat{\mathfrak{m}}^{m-n},$$

Thus it follows that $\widehat{A} \simeq \mathbb{C}[[x, y, t, z_0]]/(h, z_0 - f(t)) \simeq \mathbb{C}[[x, y, t]]/(g) = \mathcal{B}$.

Remark 1.6.5. An analogous definition of *finite definition* can be given for ideals in a polynomial ring $P = k[x_i \mid i \in I]$. It is easy to see that the definition does not depend on the choice of indeterminates, and that an ideal of P is of finite definition if and only if it is finitely generated. By contrast, in a power series ring not every ideal of finite definition is so with respect to the given indeterminates x_i, and not every finitely generated ideal is of finite definition. For instance, consider $\widehat{P} = k[[x_n \mid n \in \mathbb{N}]]$. The principal ideal generated by $f = \sum_{n\geq 1} x_n^n$ is of finite definition by Corollary 1.3.31 but not in the indeterminates x_i. As for the second claim, an example is given by the principal ideal generated by $g = \sum_{n\geq 1} x_n^{n+1}$, which, as we shall discuss next, is not of finite definition if k is of characteristic 0. Indeed, assume by contradiction that there exists an isomorphism $\widehat{P} \simeq k[[y_n \mid n \in \mathbb{N}]]$ such that $g\widehat{P}$ is of finite definition with respect to the indeterminates y_n. Pick a variable y_r not appearing in the generators for $g\widehat{P}$, and consider the regular continuous derivation $d = \partial/\partial y_r$ on $\widehat{P}$. Notice that $d(g) = 0$. By regularity, we have $d(x_m) \in \widehat{P}^\times$ for some $m \geq 1$. Writing $d(g) = \sum_{n\geq 1}(n + 1)x_n^n d(x_n)$, we see that $\mathrm{ord}_{x_m}(d(g)) < \infty$, contradicting the fact that $d(g) = 0$.

We are interested in understanding heights of ideals of finite definition. We start by looking at their minimal primes.

Proposition 1.6.6. *If $\mathfrak{a} \subset \widehat{P}$ is an ideal of finite definition, then $\mathfrak{a}$ has a finite number of minimal primes, and each of them is of finite definition. More precisely, let $J \subset I$ be a finite subset and assume that $\mathfrak{a} = \mathfrak{a}_J\widehat{P}$. If $\mathfrak{p} \subset \widehat{P}$ is a minimal prime of $\mathfrak{a}$, then $\mathfrak{p} = \mathfrak{p}_J\widehat{P}$. Moreover, the assignment $\mathfrak{p} \mapsto \mathfrak{p}_J$ gives a bijection between the minimal primes of $\mathfrak{a}$ and the minimal primes of $\mathfrak{a}_J$.*

Proof. Notice that if $\mathfrak{p} \subset \widehat{P}$ is a prime ideal, then $\mathfrak{p}_J \subset \widehat{P}_J$ remains prime. Moreover, by Corollary 1.4.8 we have that $\widehat{P}_J \to \widehat{P}$ is faithfully flat and thus

$\mathfrak{q} = (\mathfrak{q}\widehat{P}) \cap \widehat{P}_J$ for any ideal $\mathfrak{q} \subset \widehat{P}_J$. It is therefore sufficient to show that, for every prime ideal $\mathfrak{q} \subset \widehat{P}_J$, the extension $\mathfrak{q}\widehat{P}$ is prime. By Remark 1.2.2 we have an injection $\widehat{P} \to (\widehat{P}_J)^{\mathbb{N}^{(I\setminus J)}}$. Since J is finite, $\widehat{P}_J$ is Noetherian and $\mathfrak{q}$ is finitely generated. This implies that $\mathfrak{q}\,(\widehat{P}_J)^{\mathbb{N}^{(I\setminus J)}} = \mathfrak{q}^{\mathbb{N}^{(I\setminus J)}}$, that is, the elements of the extension $\mathfrak{q}\,(\widehat{P}_J)^{\mathbb{N}^{(I\setminus J)}}$ are precisely the Bourbaki power series that, when expanded in the variables indexed by $I \setminus J$, have coefficients in $\mathfrak{q}$. Therefore have an injection

$$\widehat{P}/\mathfrak{q}\widehat{P} \hookrightarrow (\widehat{P}_J/\mathfrak{q})^{\mathbb{N}^{(I\setminus J)}}$$

and the ring in the right hand side is clearly a domain. Thus $\mathfrak{q}\widehat{P}$ is prime. □

Remark 1.6.7. In the setup of the proof of Proposition 1.6.6, if J is infinite then it is no longer true that $\mathfrak{q}\,(\widehat{P}_J)^{\mathbb{N}^{(I\setminus J)}} = \mathfrak{q}^{\mathbb{N}^{(I\setminus J)}}$ for an arbitrary prime $\mathfrak{q} \subset \widehat{P}_J$. For example, let $J = \mathbb{N}$, pick $i_0 \in I \setminus J$, let $\mathfrak{q} = \widehat{\mathfrak{m}}_J$ be the maximal ideal in $\widehat{P}_J$, and consider the series $f = \sum_{n\in\mathbb{N}} x_n x_{i_0}^n$. Then f the belongs to $\mathfrak{q}^{\mathbb{N}^{(I\setminus J)}}$ but not to $\mathfrak{q}\,(\widehat{P}_J)^{\mathbb{N}^{(I\setminus J)}}$. We do not know if the extension $\mathfrak{q}\widehat{P}$ remains prime when J is infinite.

Remark 1.6.8. Proposition 1.6.6 shows that the ideal $(x_i \mid i \in J)\widehat{P}$ is prime whenever J is finite. Since colimits of prime ideals remain prime, one sees that $(x_i \mid i \in J)\widehat{P}$ is prime for an arbitrary subset J. In particular $\mathfrak{m}_0 = (x_i \mid i \in I)\widehat{P}$ is prime. Notice that $\widehat{P}/\mathfrak{m}_0$ has infinite dimension when I is infinite.

The proof of the following theorem uses the results of Section 1.4 and Proposition 1.6.12.

Theorem 1.6.9. *If $J \subset I$ is a finite subset, and $\mathfrak{a} = \mathfrak{a}_J\widehat{P}$, then* $\mathrm{ht}(\mathfrak{a}) = \mathrm{ht}(\mathfrak{a}_J)$.

Proof. From Proposition 1.6.6 we can assume that $\mathfrak{a} = \mathfrak{p} = \mathfrak{p}_J\widehat{P}$ is a prime ideal. Notice that $\mathfrak{p} \subset \mathfrak{b} := (x_j \mid j \in J)$. From Proposition 1.6.12 the localization $\widehat{P}_{\mathfrak{b}}$ is Noetherian, and therefore $\widehat{P}_{\mathfrak{p}}$, which is a further localization of $\widehat{P}_{\mathfrak{b}}$, is also Noetherian. By Corollary 1.4.8 the extension $\widehat{P}_J \subset \widehat{P}$ is flat, and therefore the extension $(\widehat{P}_J)_{\mathfrak{p}_J} \subset \widehat{P}_{\mathfrak{p}}$ is also flat. Since $\widehat{P}_{\mathfrak{p}}$ is Noetherian, it follows from by [Mat89, Theorem 15.1] that $\mathrm{ht}(\mathfrak{p}) = \mathrm{ht}(\mathfrak{p}_J)$. □

Corollary 1.6.10. *Let $\mathfrak{a} \subset \widehat{P}$ be any ideal of finite definition. For every minimal prime $\mathfrak{p}$ of $\mathfrak{a}$, we have* $\mathrm{ht}(\mathfrak{p}) < \infty$.

Remark 1.6.11. In the case of polynomial rings these properties are easy to prove, and in fact there is a strong converse to the analogue of Corollary 1.6.10, since every prime ideal of finite height in a polynomial ring $P = k[x_i \mid i \in I]$ is finitely generated. To see this, suppose $\mathfrak{p} \subset k[x_i \mid i \in I]$ is a prime ideal that is not finitely generated. Recall that $\mathfrak{p}$ is the colimit of the ideals $\mathfrak{p} \cap P_J$ as J ranges among the finite subsets of I. This implies that we can fix an embedding $\mathbb{N} \subset I$ and find an increasing sequence $\{r_n \mid n \in \mathbb{N}\} \subset \mathbb{N}$ such that, if $\mathfrak{p}_n \subset k[x_i \mid i \in I]$ is the ideal generated by $\mathfrak{p} \cap k[x_1, \dots, x_{r_n}]$, then $\mathfrak{p}_n \subsetneq \mathfrak{p}_{n+1}$ for all n. Since $\mathfrak{p}_n$ are all prime and are contained in $\mathfrak{p}$, it follows that $\mathrm{ht}(\mathfrak{p}) = \infty$.

Moreover, for arbitrary ideals $\mathfrak{a}$ of P it is proven in [GH93, Theorem 3.3] that $\mathfrak{a}$ is finitely generated if and only if it has finitely many associated primes, each of which is of finite height.

Proposition 1.6.12. *For every finite $J \subset I$, the localization $\widehat{P}_{(x_j \mid j \in J)}$ is Noetherian.*

Proof. As discussed in Remark 1.2.3, since J is finite we have an isomorphism

$$\widehat{P} \simeq k[[x_i \mid i \in I \setminus J]][[x_j \mid j \in J]].$$

The proposition now follows from the next lemma. □

Lemma 1.6.13. *For any $n \in \mathbb{N}$, let $\widehat{P}_n := \widehat{P}[[y_1, \dots, y_n]]$ and consider the ideal $\mathfrak{b}_n := (y_1, \dots, y_n)$ in $\widehat{P}_n$. Then the localization $(\widehat{P}_n)_{\mathfrak{b}_n}$ is a Noetherian ring.*

The proof of Lemma 1.6.13 uses the generalization of the Weierstrass division theorem established in Proposition 1.2.27. Recall that $f \in \widehat{P}_n$ is *y_n-regular* of order d if its image under the canonical map $\widehat{P}_n \to k[[y_n]]$ is nonzero of order d.

Theorem 1.6.14. *Let $f \in \widehat{P}_{n+1}$ be y_{n+1}-regular of order d. For every $g \in \widehat{P}_{n+1}$ there exist unique $q \in \widehat{P}_{n+1}$ and $r \in \widehat{P}_n[y_{n+1}]$ such that $g = qf + r$ and r has degree $< r$ as a polynomial in y_{n+1}.*

The next lemma ensures that we can apply Theorem 1.6.14 to prove Lemma 1.6.13.

Lemma 1.6.15. *Let $f \in \widehat{P}_{n+1} = \widehat{P}[[y_1, \dots, y_{n+1}]]$ be a nonzero element. Then there exists a continuous k-automorphism $\varphi \colon \widehat{P}_{n+1} \to \widehat{P}_{n+1}$ such that $\varphi(\mathfrak{b}_{n+1}) = \mathfrak{b}_{n+1}$ and $\varphi(f)$ is y_{n+1}-regular.*

Proof. If f is already y_{n+1}-regular, then we are done. If not, then pick any monomial of the form $x_{i_1}^{d_1} \cdots x_{i_r}^{d_r} y_1^{e_1} \cdots y_{n+1}^{e_{n+1}}$ appearing in the expansion of f. Then decompose f as

$$f = f' + f'', \quad f' \in k[[x_{i_1}, \dots, x_{i_r}, y_1, \dots, y_{n+1}]],$$

such that f' cannot be decomposed further as above. By [Art69, Lemma 6.11] there exist new coordinates $x'_{i_j} = x_{i_j} + y_{n+1}^{a_j}$, $y'_l = y_l + y_{n+1}^{b_l}$, and $y'_{n+1} := y_{n+1}$. such that $f'(x'_{i_j}, y'_l)$ is y'_{n+1}-regular. We may extend this change of coordinates trivially to a continuous automorphism $\varphi \colon \widehat{P}_{n+1} \to \widehat{P}_{n+1}$ by setting $x'_i = x_i$ for all indices i that are different from i_j for all j. Then clearly $\varphi(f)$ is y_{n+1}-regular and φ fixes $\mathfrak{b}_{n+1} = (y_1, \dots, y_{n+1})$. □

Proof of Lemma 1.6.13. We prove the lemma by induction on n. Let $Q_n := (\widehat{P}_n)_{\mathfrak{b}_n}$. Clearly $Q_0 \simeq \mathrm{Quot}(\widehat{P})$, so let us assume that Q_n is Noetherian. We have injections $Q_n \to Q_{n+1}$. Let $\mathfrak{a}$ be an ideal of Q_{n+1} and $f \in \mathfrak{a}$, $f \neq 0$. After multiplication by a unit we may assume $f \in \widehat{P}_{n+1}$; by Lemma 1.6.15 we may also assume f is y-regular. Consider the ideal $\mathfrak{a}' := \mathfrak{a} \cap Q_n[y_{n+1}]$. Since Q_n is Noetherian, so is $Q_n[y_{n+1}]$ and thus there exist $f_1, \dots, f_r \in Q_n[y_{n+1}]$ that generate $\mathfrak{a}'$. We claim that $\mathfrak{a} = (f, f_1, \dots, f_r)Q_{n+1}$.

Let $g \in \mathfrak{a}$. By Theorem 1.6.14, there exist a unit $u \in Q_{n+1}$, $q \in \widehat{P}_{n+1}$, and $r \in \widehat{P}_n[y_{n+1}]$ such that $ug = qf + r$. Since $r \in \mathfrak{a}'$, we can find $v_1, \dots, v_r \in Q_n$ such that $r = \sum_{j=1}^r v_j f_j$. Hence we have $g = u^{-1}qf + \sum_{j=1}^r u^{-1} v_j f_j$, which proves our claim. □

Chapter 2

Differentials and derivations

The main objective of this chapter is to introduce differentials and derivations as a tool both for the study of arc spaces and the local structure of singularities. Sections 2.1 and 2.2 are taken from [CN20] and are devoted to the study of higher derivations of both rings and modules. We give a construction of the respective universal objects, the Hasse-Schmidt algebra and the Hasse-Schmidt module, and recover the formula for the sheaf of differentials of jet and arc spaces in [dFD20] in the affine case (see also Section 3.2). In Section 2.3 we give an extended criterion for the existence of smooth factors in terms of regular derivations as in [CH17]. A brief digression to the case of algebraic varieties follows in Section 2.4, where we reprove a result of Ephraim (see [Eph78]) in the algebraic case. This serves as additional motivation for considering the *embedding codimension* as a formal invariant in Section 2.5, which was taken from [CdFD20].

2.1 Higher derivations and the Hasse–Schmidt algebra

Higher derivations (often also referred to as *Hasse–Schmidt derivations*) were introduced by [SH37] as an extension of the classical notion of a derivation of rings which is better behaved in the positive characteristic case. Our interest in them here is two-fold: on the one hand (continuous) higher derivations are used in the most general statement of the Zariski–Lipman–Nagata theorem (see Section 2.3); on the other hand the universal algebra of higher derivations is the algebraic counterpart of the n-th jet space resp. the arc space of an algebraic variety, which is the main object of study in Chapter 3.

In this section we will briefly recall some well-established facts about higher derivations and the universal algebra associated to them, which we call the Hasse–Schmidt algebra and denote by $\mathbb{HS}^n_k(A)$. Our main references here are [Voj07] and [Rib71a], [Rib71b], where a more in-depth treatment can be found. Furthermore, we will extend these notions to graded rings and show that the corresponding universal object can be obtained by endowing $\mathbb{HS}^n_k(A)$ with an induced (natural) grading. This grading will become useful in Section 2.2, where it will allow us to prove Theorem 2.2.12 as one of the main results of that section.

2.1.1 Higher derivations

If A, C are k-algebras and $n \in \mathbb{N} \cup \{\infty\}$, then a higher derivation $D = (D_i)_{i=0}^n : A \to C$ of order n is a collection of k-linear maps $D_i : A \to C$ such that D_0 is a map of k-algebras and the *higher Leibniz rules* are satisfied:

$$D_i(ab) = \sum_{k+l=i} D_k(a) D_l(b).$$

We write $D \in \mathrm{HS}_k^n(A, C)$. There exist bijections

$$\mathrm{HS}_k^n(A, C) \simeq \mathrm{Hom}_{\mathrm{Alg}_k}(A, C[[t]]_n), \tag{2.1a}$$

which are natural in A and C. The image of D under this map will be denoted by γ_D.

If C has in addition an A-algebra structure, then we write $\mathrm{HS}_{A/k}^n(A, C)$ for the subset of all higher derivations D such that $D_0 : A \to C$ is the structure map $a \mapsto a \cdot 1_C$. The natural bijections Eq. (2.1a) restrict to

$$\mathrm{HS}_{A/k}^n(A, C) \simeq \mathrm{Hom}^{\circ}_{\mathrm{Alg}_k}(A, C[[t]]_n), \tag{2.1b}$$

where the right-hand side denotes the subset of k-algebra maps $\gamma : A \to C[[t]]_n$ such that γ modulo (t) equals the structure map $A \to C$.

For any k-algebra A the functor $\mathrm{HS}_k^n(A, -)$ is representable by a k-algebra $\mathbb{HS}_k^n(A)$, the Hasse–Schmidt algebra, which comes equipped with a universal higher derivation $d_A^n = (d_{A,i}^n) : A \to \mathbb{HS}_k^n(A)$. The map $A \to \mathbb{HS}_k^n(A)[[t]]_n$ corresponding to d_A^n under Eq. (2.1a) will be denoted by γ_A^n. By definition for every k-algebra C there exists natural bijections

$$\mathrm{Hom}_{\mathrm{Alg}_k}(\mathbb{HS}_k^n(A), C) \simeq \mathrm{HS}_k^n(A, C) \simeq \mathrm{Hom}_{\mathrm{Alg}_k}(A, C[[t]]_n), \tag{2.1c}$$

given by

$$\varphi \in \mathrm{Hom}_{\mathrm{Alg}_k}(\mathbb{HS}_k^n(A), C) \longmapsto \widetilde{\varphi} \circ \gamma_A^n \in \mathrm{Hom}_{\mathrm{Alg}_k}(A, C[[t]]_n),$$

where $\widetilde{\varphi} : \mathbb{HS}_k^n(A)[[t]]_n \to C[[t]]_n$ is the t-linear extension of φ. We write φ_D for the map $\mathbb{HS}_k^n(A) \to C$ corresponding to a higher derivation $D \in \mathrm{HS}_k^n(A, C)$: $D_i = \varphi_D \circ d_{A,i}^n$ for all i. For convenience's sake we will often write $A_n := \mathbb{HS}_k^n(A)$.

Similarly, for every A-algebra C we obtain bijections

$$\mathrm{Hom}_{\mathrm{Alg}_A}(\mathbb{HS}_k^n(A), C) \simeq \mathrm{HS}_{A/k}^n(A, C) \simeq \mathrm{Hom}^{\circ}_{\mathrm{Alg}_k}(A, C[[t]]_n), \tag{2.1d}$$

which are natural in C.

Remark 2.1.1. The assignment $A \mapsto \mathbb{HS}_k^n(A)$ yields a functor $\mathrm{Alg}_k \to \mathrm{Alg}_k$; in particular, any k-algebra map $f : A \to A'$ gives rise to a natural map $f_n : \mathbb{HS}_k^n(A) \to \mathbb{HS}_k^n(A')$ such that $f_n \circ d_{A,i}^n = d_{A',i}^n \circ f$ for all i.

Remark 2.1.2. For $m > n$ there exist natural maps $\pi_{m,n}^* : \mathbb{HS}_k^n(A) \to \mathbb{HS}_k^m(A)$ obtained from the truncations

$$C[[t]]_m \to C[[t]]_n.$$

We will refer to the maps $\pi_{m,n}^*$ as *co-truncation maps.*

Remark 2.1.3. The co-truncation maps $\pi^*_{m,n}$ give rise to a directed system and we have $\mathbb{HS}^\infty_k(A) \simeq \operatorname{colim}_n \mathbb{HS}^n_k(A)$. Indeed, this follows directly from the universal property of the Hasse–Schmidt algebra, as we have natural bijections

$$\operatorname{Hom}_{\mathrm{Alg}_k}(A, C[[t]]) \simeq \operatorname{Hom}_{\mathrm{Alg}_k}(A, \varprojlim_n C[[t]]_n) \simeq \varprojlim_n \operatorname{Hom}_{\mathrm{Alg}_k}(A, C[[t]]_n).$$

The Hasse–Schmidt algebra $\mathbb{HS}^n_k(A)$ can be constructed as a quotient of the polynomial algebra

$$A[x^{(i)} \mid x \in A,\ i = 1, \ldots, n]$$

by the ideal generated by

$$\begin{aligned} (x+y)^{(i)} - x^{(i)} - y^{(i)},\ x, y \in A \\ (xy)^{(i)} - \sum_{k+l=i} x^{(k)} y^{(l)},\ x, y \in A \\ a^{(i)},\ a \in k, \end{aligned} \tag{2.1e}$$

for $i = 1, \ldots, m$; note that we identify $x^{(0)}$ with $x \in A$. See [Voj07] for more details. In this presentation the universal higher derivation d^n_A is given by $d^n_{A,i}(x) = x^{(i)}$.

Remark 2.1.4. There exists a natural $\mathbb{N}$-grading of $\mathbb{HS}^n_k(A)$ given by $\deg(x^{(i)}) = i$ for $x \in A$, which we will refer to as the *structural grading* of $\mathbb{HS}^n_k(A)$. Indeed, notice that the system of equations (2.1e) is homogeneous with respect to $\deg(x^{(i)}) = i$. Moreover, for any k-algebra map $f : A \to A'$ the natural map $f_n : \mathbb{HS}^n_k(A) \to \mathbb{HS}^n_k(A')$ from Remark 2.1.1 is graded.

2.1.2 Graded higher derivations

Let us first consider the case $n \in \mathbb{N}$, i.e. of higher derivations of finite order. Let k be a ring and $A = \bigoplus_{i \in \mathbb{N}} A_i$ and $C = \bigoplus_{i \in \mathbb{N}} C_i$ graded k-algebras (where we regarded k with the trivial grading; in particular the sets of homogeneous elements A_i and C_i are k-modules). We call a higher derivation $D = (D_i)_i : A \to C$ of order n *graded* if every component D_i is graded (of degree 0), that is, we have $D_i(A_j) \subset C_j$ for all $j \in \mathbb{N}$. The set of all such D will be denoted by $\mathrm{HS}^n_{k,\mathrm{gr}}(A, C)$. Note that $C[[t]]_n$ is a free C-module of rank $n+1$ and thus carries a natural grading induced by C. It is then immediate that the natural isomorphism Eq. (2.1a) restricts to

$$\mathrm{HS}^n_{k,\mathrm{gr}}(A, C) \simeq \operatorname{Hom}_{\mathrm{Alg}_{k,\mathrm{gr}}}(A, C[[t]]_n).$$

We claim that there exists an $\mathbb{N}$-grading on $\mathbb{HS}^n_k(A)$, different from the structural grading introduced in Remark 2.1.4, such that, for every $\mathbb{N}$-graded k-algebra C, the natural bijections Eq. (2.1c) restrict to natural bijections

$$\operatorname{Hom}_{\mathrm{Alg}_{k,\mathrm{gr}}}(\mathbb{HS}^n_k(A), C) \simeq \mathrm{HS}^n_{k,\mathrm{gr}}(A, C) \simeq \operatorname{Hom}_{\mathrm{Alg}_{k,\mathrm{gr}}}(A, C[[t]]_n).$$

In particular $\gamma_A : A \to \mathbb{HS}^n_k(A)[[t]]_n$ will be a map of graded k-algebras. We call this grading the *induced grading* of $\mathbb{HS}^n_k(A)$. To construct the induced grading, note that, if $A = \bigoplus_{i \in \mathbb{N}} A_i$, then the presentation given by Eq. (2.1e) can be refined such that A is given as the quotient of

$$A[x^{(i)} \mid x \in A_j,\ j \in \mathbb{N},\ i = 1, \ldots, n]$$

by the ideal generated by

$$(x+y)^{(i)} - x^{(i)} - y^{(i)},\ x,y \in A_j \tag{2.1f}$$
$$(xy)^{(i)} - \sum_{k+l=i} x^{(k)}y^{(l)},\ x,y \in A_j$$
$$a^{(i)},\ a \in k,$$

for $i = 1, \ldots, m$ and $j \in \mathbb{N}$. We define the induced grading by setting $\deg(x^{(i)}) := j$ for $x \in A_j$. Note that the system Eq. (2.1f) is homogeneous with respect to this grading, so it is well-defined. The k-module $(\mathbb{HS}^n_k(A))_i$ of elements of degree i is generated by the set of products

$$\{x_1^{(j_1)} \cdots x_r^{(j_r)} \mid x_l \in A_{i_l},\ i_1 + \ldots + i_r = i\}.$$

Note that $(\mathbb{HS}^n_k(A))_0 = \mathbb{HS}^n_k(A_0)$ and $(\mathbb{HS}^n_k(A))_i$ is a module over $\mathbb{HS}^n_k(A_0)$.

Now to see the claim above, if $\varphi_D : \mathbb{HS}^n_k(A) \to C$ is graded, then $\varphi_D(x^{(j)}) \in C_i$ for $x \in A_i$. The map $\gamma_D : A \to C[[t]]_n$ corresponding to φ_D under Eq. (2.1c) is given by

$$x \mapsto \sum_{j=0}^{n} \varphi_D(x^{(j)})t^j,$$

and thus is graded. The other direction follows in analogy.

Remark 2.1.5. Under the above hypotheses, for any $D \in \mathrm{HS}^n_{k,\mathrm{gr}}(A, C)$, we consider $D^0 = (D^0_i)_i$, with $D^0_i : A_0 \to C_0$ the degree 0 part of $D_i : A \to C$. It is clear that $D^0 \in \mathrm{HS}^n_k(A_0, C_0)$.

In fact, taking the Hasse–Schmidt algebra gives rise to a functor $\mathrm{Alg}_{k,\mathrm{gr}} \to \mathrm{Alg}_{k,\mathrm{gr}}$:

Theorem 2.1.6. *Let $n \in \mathbb{N}$ and $A = \bigoplus_i A_i$ be a graded k-algebra (where we consider k with the trivial grading). Then the assignment $A \mapsto \mathbb{HS}^n_k(A)$ yields a functor $\mathrm{Alg}_{k,\mathrm{gr}} \to \mathrm{Alg}_{k,\mathrm{gr}}$ satisfying the following: for every $\mathbb{N}$-graded k-algebra C there exist bijections*

$$\mathrm{Hom}_{\mathrm{Alg}_{k,\mathrm{gr}}}(\mathbb{HS}^n_k(A), C) \simeq \mathrm{Hom}_{\mathrm{Alg}_{k,\mathrm{gr}}}(A, C[[t]]_n), \tag{2.1g}$$

natural in A and C.

Remark 2.1.7. Similarly, if C is a graded A-algebra and $\mathrm{HS}^n_{A/k,\mathrm{gr}}(A, C)$ denotes the subset of graded higher derivations $D : A \to C$ such that $D_0 : A \to C$ is the structure map, then there exist natural bijections

$$\mathrm{Hom}_{\mathrm{Alg}_{A,\mathrm{gr}}}(\mathbb{HS}^n_k(A), C) \simeq \mathrm{HS}^n_{A/k,\mathrm{gr}}(A, C) \simeq \mathrm{Hom}^\circ_{\mathrm{Alg}_{k,\mathrm{gr}}}(A, C[[t]]_n). \tag{2.1h}$$

Note that $\mathbb{HS}^n_k(A)$ is a graded A-algebra by construction.

Let us now consider the case $n = \infty$. Clearly the co-truncation maps $\pi^*_{m,n} : \mathbb{HS}^n_k(A) \to \mathbb{HS}^m_k(A)$, $m > n$ preserve the gradings on both sides. By Remark 2.1.3 we obtain a grading on $\mathbb{HS}^\infty_k(A)$ coming from its direct limit structure. However, note that there exists no result which is directly analogous to Theorem 2.1.6, since, for C a graded k-algebra, the power series ring $C[[t]]$ does no longer carry a grading compatible with that of C itself. In fact, one would need to interpret $\mathrm{Hom}_{\mathrm{Alg}_{k,\mathrm{gr}}}(A, C[[t]])$ as the set of all k-algebra maps $\varphi : A \to C[[t]]$ such that $\varphi(A_i) \subset C_i[[t]]$ for all $i \geq 0$.

Remark 2.1.8. The structural grading of $\mathbb{HS}_k^n(A)$ (see Remark 2.1.4) together with its induced grading yields an $\mathbb{N}$-bigrading of $\mathbb{HS}_k^n(A)$. Indeed, observe that in the presentation given by Eq. (2.1f) this bigrading is defined via $\deg(x^{(j)}) = (i,j)$ for $x \in A_i$. Moreover, for $f : A \to A'$ a map between graded k-algebras, the induced map $f_n : \mathbb{HS}_k^n(A) \to \mathbb{HS}_k^n(A')$ clearly respects the bigrading. Hence we can view $A \mapsto \mathbb{HS}_k^n(A)$ as a functor from $\mathbb{N}$-graded k-algebras to $\mathbb{N}$-bigraded k-algebras.

The following lemma states that the functors $\mathbb{HS}_k^n(-)$ commute with each other and is one of the main ingredients of the proof of Theorem 2.2.15.

Lemma 2.1.9. *Let A be a k-algebra. For all $m, n \in \mathbb{N} \cup \{\infty\}$ there exist natural isomorphisms of graded A_n-algebras*

$$\mathbb{HS}_k^n(\mathbb{HS}_k^m(A)) \simeq \mathbb{HS}_k^m(\mathbb{HS}_k^n(A)),$$

where we consider the left-hand side with its structural grading and the right-hand side with its induced grading.

Remark 2.1.10. An alternative way of stating Lemma 2.1.9 is by means of *multivariate* higher derivations. See [Nar18, Corollary 2.3.12] for a precise statement and [Nar20, Section 2] for generalities on multivariate higher derivations.

Proof of Lemma 2.1.9. We give a short sketch of the proof here. Assume first that $n \in \mathbb{N}$ and C is a graded A_n-algebra. Consider the chain of isomorphisms

$$\begin{aligned} &\operatorname{Hom}_k(\mathbb{HS}_k^n(\mathbb{HS}_k^m(A)), C) \simeq \operatorname{Hom}_k(A, (C[[s]]_n)[[t]]_m) \\ &\simeq \operatorname{Hom}_k(A, (C[[t]]_m)[[s]]_n) \simeq \operatorname{Hom}_k(\mathbb{HS}_k^m(\mathbb{HS}_k^n(A)), C). \end{aligned} \tag{2.1i}$$

Then it is straightforward to check that all isomorphisms in Eq. (2.1i) preserve both the grading and the A_n-algebra structure. Thus we are done by Theorem 2.1.6. For the case $n = \infty$, observe that the functor $A \mapsto \mathbb{HS}_k^m(A)$ is a left adjoint and therefore the statement follows from

$$\operatorname*{colim}_n \mathbb{HS}_k^n(\mathbb{HS}_k^m(A)) \simeq \operatorname*{colim}_n \mathbb{HS}_k^m(\mathbb{HS}_k^n(A)) \simeq \mathbb{HS}_k^m(\operatorname*{colim}_n \mathbb{HS}_k^n(A)).$$

□

2.2 Higher derivations of modules and the Hasse–Schmidt module

Higher derivations of modules were first introduced in [Rib80] in analogy to higher derivations of rings as a means of providing a notion of a module homomorphism carrying “infinitesimal information”. In particular, there too exists a universal object parametrizing such higher derivations of modules, which we call the *Hasse–Schmidt module*. This construction was implicitly considered in [dFD20] when establishing a formula for the sheaf of differentials of jet and arc spaces (the full version of which we will prove in Section 3.2). Their main observation was that the Hasse–Schmidt module is given by tensoring with a module Q_n, which we construct here as the dual of a certain bimodule. As such, we start this section with a brief digression on the notion of a (strongly) *dualizable* bimodule, which is a natural extension of the notion of a dualizable object in a monoidal category. We will then recall the definition of a higher derivation of modules and the Hasse–Schmidt module and compare the latter to the Hasse–Schmidt algebra by means of the symmetric algebra (see Theorem 2.2.12). Finally, we will prove the main formula in [dFD20] in the affine case, using Theorem 2.2.12 and the fact that the Hasse–Schmidt algebra functors commute. We should also mention that Section 3.2 should be read as a natural continuation of this section, providing a global version of the results here after introducing jet and arc spaces.

2.2.1 Dualizable bimodules and limits

Let A be a ring and M a left A-module, then the right A-module $M^\vee := \mathrm{Hom}_A(M, A)$ is commonly called the *dual module* of M. In general, the dual of a module is not well-behaved, as for example M might not be *reflexive*, i.e. the natural map $M \to (M^\vee)^\vee$ might not be an isomorphism. For a module M to be *dualizable* we will thus require it to have a dual object in the (stronger) categorical sense of [DP80]. In this section we will recall some well-known and elementary facts about dualizable (bi)modules, which will be used to establish the existence of the Hasse–Schmidt module in Section 2.2.

Let us start by fixing some notation. If M and N are both right A-modules, we denote the set of right A-homomorphisms $M \to N$ by $\mathrm{Hom}_{(-,A)}(M, N)$. Similarly, if M and N are left B-modules, we write $\mathrm{Hom}_{(B,-)}(M, N)$ for the set of left B-homomorphisms $M \to N$. This choice of notation will allow us to be precise about the type of structure considered when dealing with bimodules.

Let A, B be (not necessarily) commutative rings. Recall that a (B, A)-bimodule P is an abelian group which is both a left B-module and a right A-module and which satisfies $b(xa) = (bx)a$ for $x \in P$, $a \in A$, $b \in B$. We say that P is *left dualizable* if there exists an (A, B)-bimodule Q and bimodule homomorphisms

$$\eta : A \to Q \otimes_B P,\ \theta : P \otimes_A Q \to B$$

such that the diagrams

$$\begin{tikzcd}
A \otimes_A Q \arrow[r, "\eta\otimes \mathrm{id}_Q"] \arrow[d, "\simeq"] & (Q \otimes_B P) \otimes_A Q \arrow[d, "\simeq"] \\
Q \otimes_B B & Q \otimes_B (P \otimes_A Q) \arrow[l, "\mathrm{id}_Q \otimes \theta"']
\end{tikzcd}$$

and

$$\begin{tikzcd}
P \otimes_A A \arrow[r, "\mathrm{id}_P \otimes \eta"] \arrow[d, "\simeq"] & P \otimes_A (Q \otimes_B P) \arrow[d, "\simeq"] \\
B \otimes_B P & (P \otimes_A Q) \otimes_B P, \arrow[l, "\theta\otimes \mathrm{id}_P"']
\end{tikzcd}$$

commute, with the vertical arrows being the associators and unitors. The map θ is called the *evaluation* and the map η the *coevaluation*. Furthermore, the (A, B)-bimodule Q is unique up to isomorphism and will be referred to as the *left dual* of P. If there is no ambiguity we will write $P^\vee = Q$ and $Q^\vee = P$. Note that P being left dualizable is equivalent to saying that the functor $- \otimes_B P$ from right B-modules to right A-modules is right adjoint to $- \otimes_A Q$; in particular, by tensor-hom adjunction we have a natural isomorphism

$$P \simeq \mathrm{Hom}_{(-,B)}(Q, B).$$

We will now recall this classical result, which characterizes dualizable bimodules as those who are finite projective.

Theorem 2.2.1. *Let P be a (B, A)-bimodule. Then P is left dualizable if and only if it is finitely generated projective as a left B-module. In particular, we have bijections*

$$\mathrm{Hom}_{(-,A)}(M, N \otimes_B P) \simeq \mathrm{Hom}_{(-,B)}(M \otimes_A P^\vee, N),$$

which are natural in M and N.

Remark 2.2.2. If a functor G is left adjoint to $-\otimes_B P$, then G is automatically of the form $-\otimes_A Q$ for some (A, B)-bimodule Q (see for example [Sch09, Theorem 3.60]). Thus P being left dualizable is equivalent to $- \otimes_B P$ being right adjoint.

Remark 2.2.3. If $\mathcal{C}$ is a monoidal category (for example, the category of modules over a commutative ring R), then a dualizable object X of $\mathcal{C}$ is an adjoint to the morphism corresponding to X in the delooping 2-category $\mathbf{B}\mathcal{C}$. Similarly, a bimodule P is left dualizable if the corresponding 1-morphism in the bicategory of algebras, bimodules and intertwiners (as introduced in [Bén67]) is a right adjoint.

For the purposes of the next section we need to consider the case where P is a cofiltered limit of (B, A)-bimodules P_i, which are finitely generated projective as left B-modules. In general, P will not be finite projective over B itself, so there does not exist a (left) dual of P in the sense discussed above. However, the functor $\lim_i(- \otimes_B P_i)$ does have a left adjoint, which just follows from the fact that right adjoints commute with limits. We summarize this fact in the following corollary:

Corollary 2.2.4. *Let $P = \lim_i P_i$ be a cofiltered limit of (B, A)-bimodules P_i which are finitely generated projective over B. Then the left duals $P_i^\vee$ form a filtered system. Furthermore, we have bijections*

$$\mathrm{Hom}_{(-,A)}(M, \lim_i N \otimes_B P_i) \simeq \mathrm{Hom}_{(-,B)}(M \otimes_A \operatorname*{colim}_i P_i^\vee, N),$$

which are natural in M and N.

2.2.2 Higher derivations of modules and the Hasse–Schmidt module

Higher derivations of modules were first introduced in [Rib80], where they were defined with respect to a given higher derivation of rings. The existence of a universal object, which we call Hasse–Schmidt module, parametrizing such higher derivations was already established there; however, it also appeared implicitly in more detail in [dFD20]. In this section our aim is to provide a top-down view of the construction of the Hasse–Schmidt module and how it relates to the Hasse–Schmidt algebra by means of Theorem 2.2.12, which is the main result of this section.

Throughout this section we will write $A_n := \mathbb{HS}^n_k(A)$ for the Hasse–Schmidt algebra of a k-algebra A. All rings considered here are assumed to be commutative; in particular, we do not have to distinguish between left and right actions.

Let now A and C be k-algebras and $D = (D_i)_{i=0}^n : A \to C$ be a higher derivation of length n over k. A *higher derivation over* D, $\Delta = (\Delta_i)_{i=0}^n : M \to N$, where $M \in \mathrm{Mod}_A$, $N \in \mathrm{Mod}_C$, is a collection of k-linear maps $\Delta_i : M \to N$ satisfying

$$\Delta_i(a \cdot m) = \sum_{k+l=i} D_k(a) \cdot \Delta_l(m),\ a \in A, m \in M.$$

The set of all such maps is denoted by $\mathrm{HS}^n_D(M, N)$. We have isomorphisms

$$\mathrm{HS}^n_D(M, N) \simeq \mathrm{Hom}_A(M, N[[t]]_n),$$

which are natural in M and N. Note that the A-module structure on $N[[t]]_n$ above comes from scalar restriction through $\gamma_D : A \to C[[t]]_n$, with $D_i = \varphi_D \circ d^n_{A,i}$. If we consider N as an A_n-module via restriction by $\varphi_D : A_n \to C$, then we see that

$$\mathrm{HS}^n_D(M, N) \simeq \mathrm{HS}^n_{d^n_A}(M, N).$$

Let us first treat the case $n \in \mathbb{N}$. We consider $A_n[[t]]_n$ as a (A_n, A)-bimodule, where the left action is just the usual multiplication and the right action is induced by γ_A. Then we have an isomorphism of right A-modules

$$N[[t]]_n \simeq N \otimes_{A_n} A_n[[t]]_n.$$

Note that $A_n[[t]]_n$ is free of rank $n+1$ over A_n. By Theorem 2.2.1 we get

$$\mathrm{HS}^n_{d^n_A}(M, N) \simeq \mathrm{Hom}_A(M, N \otimes_{A_n} A_n[[t]]_n) \simeq \mathrm{Hom}_{A_n}(M \otimes_A (A_n[[t]]_n)^\vee, N),$$

where $(A_n[[t]]_n)^\vee \simeq \mathrm{Hom}_{(-,A_n)}(A_n[[t]]_n, A_n)$ is the left dual of $A_n[[t]]_n$. Using the fact that extension of scalars is left adjoint to restriction, we obtain

$$\mathrm{HS}^n_D(M, N) \simeq \mathrm{Hom}_C(M \otimes_A (A_n[[t]]_n)^\vee \otimes_{A_n} C, N)$$

Now, in the case $n = \infty$, we have that $A_\infty[[t]] = \lim_n A_\infty[[t]]_n$ is the limit of a projective system of (A_∞, A)-bimodules which are finite free over A_∞. Arguing as before and applying Corollary 2.2.4 yields

$$\mathrm{HS}^\infty_D(M, N) \simeq \mathrm{Hom}_C(M \otimes_A \operatorname*{colim}_n (A_\infty[[t]]_n)^\vee \otimes_{A_\infty} C, N).$$

Thus we have proven the following result:

Theorem 2.2.5 ([Rib80, §4])**.** *Let A and C be k-algebras and $D : A \to C$ be a higher derivation of order $n \in \mathbb{N} \cup \{\infty\}$. Then, for any A-module M, the functor*

$$\mathrm{HS}^n_D(M, -) : \mathrm{Mod}_C \to \mathrm{Set}$$

is representable by a module $\mathbb{HS}^n_{A/k}(M) \otimes_{A_n} C$, where we call the A_n-module $\mathbb{HS}^n_{A/k}(M)$ the n-th Hasse–Schmidt module *of M. Moreover, we have*

$$\mathbb{HS}^n_{A/k}(M) \simeq M \otimes_A (A_n[[t]]_n)^\vee$$

for $n \in \mathbb{N}$ and

$$\mathbb{HS}^\infty_{A/k}(M) \simeq M \otimes_A \operatorname*{colim}_n (A_\infty[[t]]_n)^\vee,$$

where $(A_\infty[[t]]_n)^\vee$ is the (left) dual of the finite free A_∞-module $A_\infty[[t]]_n$.

Remark 2.2.6. The module $\mathbb{HS}^n_{A/k}(M)$ comes attached with a universal higher derivation $\Delta_M : M \to \mathbb{HS}^n_{A/k}(M)$ over $d^n_A : A \to A_n$. We want to give an explicit description of Δ_M in the case where $M = A$, i.e. $\mathbb{HS}^n_{A/k}(M) = (A_n[[t]]_n)^\vee$. To that avail, if $n \in \mathbb{N}$, observe that the A-module map $\alpha_D : A \to \mathbb{HS}^n_{A/k}(A)[[t]]_n$ corresponding to Δ_A is just the coevaluation. Thus, if we take the standard basis t^i, $i = 0, \dots, n$ for $A_n[[t]]_n$ over A_n and let $t^{[i]}$, $i = 0, \dots, n$, denote the dual basis, the map α_A is given by

$$1_A \mapsto \sum_{i=0}^{n} t^{[i]} t^i. \tag{2.2a}$$

Note that if $n = \infty$ then the images of $t^{[i]}$, $i \in \mathbb{N}$, in $\operatorname{colim}_n(A_\infty[[t]]_n)^\vee$ form a basis over A_∞. Hence the universal higher derivation $\Delta_A : A \to \mathbb{HS}^n_{A/k}(A)$ is given by

$$(\Delta_A)_i(1_A) = t^{[i]}. \tag{2.2b}$$

Remark 2.2.7. For $n \in \mathbb{N}$, since $A_n[[t]]_n$ is finite free over A_n, we have that

$$\mathbb{HS}^n_{A/k}(M) \simeq M \otimes_A A_n[[t]]_n$$

as A_n-modules. Indeed, choosing the same basis as in Remark 2.2.6, this isomorphism is given identifying $A_n[[t]]_n$ with its dual via $t^i \mapsto t^{[i]}$.

Remark 2.2.8. As $A_n[[t]]_n$ is free for $n \in \mathbb{N} \cup \{\infty\}$, if M is projective respectively free then so is $\mathbb{HS}^n_{A/k}(M)$.

Remark 2.2.9. Alternatively one could consider triples (C, D, N), with C a k-algebra, $D \in \mathrm{HS}^n_k(A, C)$ and N a C-modules, a suitable notion of morphism between such triples and the functor which associates to each such triple the set $\mathrm{HS}^n_D(M, N)$. Then the argument given before shows that this functor is represented by the triple $(A_n, d^n_A, \mathbb{HS}^n_{A/k}(M))$.

Remark 2.2.10. We have natural isomorphisms $\mathbb{HS}^\infty_{A/k}(M) \simeq \operatorname{colim}_n \mathbb{HS}^n_{A/k}(M)$. Indeed, this follows just as in Remark 2.1.3 from

$$\mathrm{Hom}_A(M, N[[t]]) \simeq \mathrm{Hom}_A(M, \lim_n N[[t]]_n) \simeq \lim_n \mathrm{Hom}_A(M, N[[t]]_n),$$

where N is an A_n-module.

We will refer to [Rib80] for a basic introduction to the theory of higher derivations of modules. As one fact not included there, let us mention that the Hasse–Schmidt module behaves well under base change.

Lemma 2.2.11. *Let $A \to A'$ be a k-algebra map and M an A-module. Then we have*

$$\mathbb{HS}^n_{A'/k}(M \otimes_A A') \simeq \mathbb{HS}^n_{A/k}(M) \otimes_{A_n} A'_n.$$

Proof. Follows immediately from the description in Theorem 2.2.5. ☐

The following theorem is the main result of this section and relates the Hasse–Schmidt module to the Hasse–Schmidt algebra by means of the symmetric algebra.

Theorem 2.2.12. *Let A be a k-algebra and M an A-module. For each $n \in \mathbb{N} \cup \{\infty\}$ there are isomorphisms of $\mathbb{N}$-graded A_n-algebras*

$$\mathrm{Sym}_{A_n}(\mathbb{HS}^n_{A/k}(M)) \simeq \mathbb{HS}^n_k(\mathrm{Sym}_A(M)),$$

which are natural in M. Note that on the left side we are considering the natural grading on the symmetric algebra, while on the right hand side we are taking the induced grading of $\mathbb{HS}^n_k(\mathrm{Sym}_A(M))$.

Remark 2.2.13. By considering $\mathbb{HS}^n_{A/k}(M)$ as a graded A_n-module similar to Remark 2.1.4 the isomorphism in Theorem 2.2.12 extends to one of $\mathbb{N}$-bigraded algebras.

Proof. Consider first the case $n \in \mathbb{N}$ and let $B = \bigoplus_{i \geq 0} B_i$ be an $\mathbb{N}$-graded A_n-algebra, where we consider A_n with the trivial grading. Then each B_i is in particular an A_n-module. Recall that $B[[t]]_n$ is a free B-module of rank $n+1$ and thus has a natural $\mathbb{N}$-grading given by $(B[[t]]_n)_i := B_i[[t]]_n$. We obtain natural bijections

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Alg}^{\mathrm{gr}}_{A_n}}(\mathrm{Sym}_{A_n}(\mathbb{HS}^n_{A/k}(M)), B) &\simeq \mathrm{Hom}_{\mathrm{Mod}_{A_n}}(\mathbb{HS}^n_{A/k}(M), B_1) \simeq \\ &\simeq \mathrm{Hom}_{\mathrm{Mod}_A}(M, B_1[[t]]_n) \simeq \mathrm{Hom}_{\mathrm{Alg}^{\mathrm{gr}}_A}(\mathrm{Sym}_A(M), B[[t]]_n). \end{aligned}$$

Here again we consider A with the trivial grading. If $\rho : A_n \to B$ is the map defining the A_n-algebra structure on B, then the A-algebra (resp. A-module) structure on $B[[t]]_n$ (resp. $B_1[[t]]_n$) is given by $\tilde{\rho} \circ d^n_A$, where $\tilde{\rho} : A_n[[t]] \to B[[t]]$ is obtained from ρ by t-linear extension. We claim that

$$\mathrm{Hom}_{\mathrm{Alg}^{\mathrm{gr}}_A}(\mathrm{Sym}_A(M), B[[t]]_n) \simeq \mathrm{Hom}_{\mathrm{Alg}^{\mathrm{gr}}_{A_n}}(\mathbb{HS}^n_k(\mathrm{Sym}_A(M)), B).$$

Indeed, an element α of the left-hand side is given by a triangle

$$\begin{tikzcd} \mathrm{Sym}_A(M) \arrow[r, "\alpha"] & B[[t]]_n \\ A \arrow[u] \arrow[ur, "\tilde{\rho} \circ d^n_A"'] & \end{tikzcd}$$

where $\alpha = (\alpha_i)_{i \in \mathbb{N}}$ is graded of degree 0. By Theorem 2.1.6 we obtain a triangle of k-algebra maps

$$\begin{tikzcd} \mathbb{HS}^n_k(\mathrm{Sym}_A(M)) \arrow[r, "\alpha^*"] & B \ , \\ \mathbb{HS}^n_k(A) \arrow[u] \arrow[ur, "\rho"'] & \end{tikzcd}$$

where α^* and ρ is graded. Conversely, every such triangle is obtained by one of the form above. This proves the claim for $n \in \mathbb{N}$.

We are thus left with the case $n = \infty$. Taking colimits of the isomorphism for finite n we obtain

$$\operatorname*{colim}_n \mathbb{HS}^n_k(\mathrm{Sym}_A(M)) \simeq \operatorname*{colim}_n \mathrm{Sym}_{A_n}(\mathbb{HS}^n_{A/k}(M)) \simeq \mathrm{Sym}_{A_\infty}(\operatorname*{colim}_n \mathbb{HS}^n_{A/k}(M)).$$

As we have both $\mathbb{HS}^\infty_k(B) = \mathrm{colim}_n\, \mathbb{HS}^n_k(B)$ and $\mathbb{HS}^\infty_{A/k}(M) = \mathrm{colim}_n\, \mathbb{HS}^n_{A/k}(M)$ the result follows. $\square$

Remark 2.2.14. Applying Theorem 2.2.5 as well as Theorem 2.2.12 in the case $M = A$ and $n \in \mathbb{N}$ yields the following:

$$\mathbb{HS}^n_{A/k}(A) \simeq (A_n[[t]]_n)^\vee \simeq [\mathbb{HS}^n_k(\mathrm{Sym}_A(A))]_1. \tag{2.2c}$$

To make this isomorphism explicit, let $e := 1_A$. Then

$$\mathbb{HS}^n_k(\mathrm{Sym}_A(A)) \simeq A_n[e^{(i)} \mid i = 0, \ldots, n].$$

Thus the A_n-submodule of elements of degree 1 is generated by $e^{(i)}$, $i = 0, \ldots, n$. Using the same basis as in Remark 2.2.6 we see that the isomorphism in Eq. (2.2c) is given by

$$t^{[i]} \mapsto e^{(i)}.$$

Now, since the A-action on $\mathbb{HS}^n_{A/k}(A) \simeq (A_n[[t]]_n)^\vee$ is given by precomposition, we obtain an induced A-action on $[\mathbb{HS}^n_k(\mathrm{Sym}_A(A))]_1$ given by

$$a \cdot e^{(i)} = \sum_{j=0}^{i} a^{(i-j)} e^{(j)}.$$

Compare this with the construction of P_n given in [dFD20, Section 4]. It is not clear how to obtain this A-action naturally by just considering $\mathbb{HS}^n_k(\mathrm{Sym}_A(A))$.

Using Theorem 2.2.12 we will recover the formula in [dFD20, Theorem 5.3] as a direct consequence of the fact that the Hasse–Schmidt algebra functors commute (see Lemma 2.1.9).

Theorem 2.2.15. *For all $n \in \mathbb{N} \cup \{\infty\}$ there is a natural isomorphism*

$$\Omega_{A_n/k} \simeq \mathbb{HS}^n_{A/k}(\Omega_{A/k}) \simeq \Omega_{A/k} \otimes_A \mathbb{HS}^n_{A/k}(A).$$

of A_n-modules. Additionally, if $n \in \mathbb{N}$, we have

$$\Omega_{A_n/k} \simeq \Omega_{A/k} \otimes_A A_n[[t]]_n.$$

Proof. Note that $\mathrm{Sym}_B(\Omega_{B/k}) \simeq \mathbb{HS}^1_k(B)$ for any k-algebra B. Thus Lemma 2.1.9 implies that we have isomorphisms of graded algebras

$$\mathbb{HS}^n_k(\mathrm{Sym}_A(\Omega_{A/k})) \simeq \mathbb{HS}^n_k(\mathbb{HS}^1_k(A)) \simeq \mathbb{HS}^1_k(\mathbb{HS}^n_k(A)) \simeq \mathrm{Sym}_{A_n}(\Omega_{A_n/k}).$$

Thus the statement follows from Theorem 2.2.12 for $M = \Omega_{A/k}$ and taking the homogeneous part of degree 1 on each side. The second assertion just follows from Remark 2.2.7. $\square$

Remark 2.2.16. The idea of the proof gives a very clear interpretation of the formula in Theorem 2.2.15: as the n-th Hasse–Schmidt algebra $\mathbb{HS}^n_k(A)$ parameterizes infinitesimal data on A up to order n (i.e. *n-jets* on A), the formula says that tangents (i.e. infinitesimal data up to order 1) of n-jets on A can be recovered from some higher order operation on tangents on A. Indeed, this is expressed in a precise way with the fact that the Hasse–Schmidt algebra functors commute - what is left is Theorem 2.2.12, which shows that the Hasse–Schmidt modules is a linearisation of sorts of the Hasse–Schmidt algebra.

Remark 2.2.17. A similar argument allows us to consider (usual) k-derivations $d : A \to M$ with M an A-module as higher derivations in the above sense. To that end, note that

$$\mathrm{Der}_k(A, M) \simeq \mathrm{Hom}_A(\Omega_{A/k}, M) \simeq \mathrm{Hom}_{\mathrm{Alg}_A,\mathrm{gr}}(\mathrm{Sym}_A(\Omega_{A/k}), \mathrm{Sym}_A(M)).$$

Since $\mathrm{Sym}_A(\Omega_{A/k}) \simeq \mathbb{HS}^1_k(A)$ as graded k-algebras, by Remark 2.1.7 we obtain

$$\mathrm{Der}_k(A, M) \simeq \mathrm{Hom}^{\circ}_{\mathrm{Alg}_{k,\mathrm{gr}}}(A, \mathrm{Sym}_A(M)[[t]]_1).$$

Since every such map factors as $A \to \mathbb{HS}^1_k(A)[[t]]_1 \to \mathrm{Sym}_A(M)[[t]]_1$ we have

$$\mathrm{Hom}^{\circ}_{\mathrm{Alg}_{k,\mathrm{gr}}}(A, \mathrm{Sym}_A(M)[[t]]_1) \hookrightarrow \mathrm{Hom}_{\mathrm{Mod}_A}(A, \mathrm{Sym}_A(M)[[t]]_1),$$

where the A-action on the $\mathrm{Sym}_A(M)[[t]]_1$ is induced by d^1_A. Thus we obtain our claim that

$$\mathrm{Der}_k(A, M) \hookrightarrow \mathrm{HS}^1_{d^1_A}(A, \mathrm{Sym}_A(M)).$$

Remark 2.2.18. In a similar vein to Theorem 2.2.12, higher derivations of modules can be seen as graded higher derivations of symmetric algebras. Namely, if A and C are k-algebras, $M \in \mathrm{Mod}_A$, $N \in \mathrm{Mod}_C$, $D \in \mathrm{HS}^n_k(A, C)$ and $n \in \mathbb{N}$, we have:

$$\begin{aligned}
\mathrm{HS}^n_D(M, N) &\simeq \mathrm{Hom}_A(M, N[[t]]_n) \simeq \mathrm{Hom}_{\mathrm{Alg}_A,\mathrm{gr}}(\mathrm{Sym}_A(M), \mathrm{Sym}_{C[[t]]_n}(N[[t]]_n)) \\
&\simeq \mathrm{Hom}_{\mathrm{Alg}_A,\mathrm{gr}}(\mathrm{Sym}_A(M), \mathrm{Sym}_C(N)[[t]]_n) \subset \mathrm{Hom}_{\mathrm{Alg}_k,\mathrm{gr}}(\mathrm{Sym}_A(M), \mathrm{Sym}_C(N)[[t]]_n) \\
&\simeq \mathrm{HS}^n_{k,\mathrm{gr}}(\mathrm{Sym}_A(M), \mathrm{Sym}_C(N)),
\end{aligned}$$

and so any $\Delta \in \mathrm{HS}^n_D(M, N)$ gives rise to a well-defined graded higher derivation $\boldsymbol{\Delta} \in \mathrm{HS}^n_{k,\mathrm{gr}}(\mathrm{Sym}_A(M), \mathrm{Sym}_C(N))$. Notice that for the degree 0 part we have $\boldsymbol{\Delta}^0 = D$. Actually, the above procedure gives natural bijections

$$\{(D, \Delta) \mid D \in \mathrm{HS}^n_k(A, C), \Delta \in \mathrm{HS}^n_D(M, N)\} \simeq \mathrm{HS}^n_{k,\mathrm{gr}}(\mathrm{Sym}_A(M), \mathrm{Sym}_C(N)).$$

Since these bijections are compatible with truncations, they also hold for $n = \infty$.

2.3 The Zariski–Lipman–Nagata criterion and cancellation

In this section our goal is to prove a generalization of the classical Zariski–Lipman–Nagata criterion to non-Noetherian, non-adic rings and use it to give a result on cancellation in the category of formal schemes. The original statement can be found in e.g. [Zar65, Lemma 4] and gives a criterion for a complete local ring to be isomorphic to a power series ring in terms of the existence of a regular derivation. To extend this result to positive characteristics we will use higher derivations in place of usual derivations. The cancellation result we will derive from it allows us to define the *minimal formal model* of k-rational points on schemes over k. The minimal formal model will be studied further in Section 2.4 for schemes of finite type, and in Sections 3.4 and 3.5 in the context of arc spaces.

2.3.1 The Zariski–Lipman–Nagata criterion

Throughout this section we assume k to be a commutative ring (which we will consider as a topological ring by endowing it with the discrete topology). As in previous section, for A any ring and $n \in \mathbb{N} \cup \{\infty\}$ we write $A[[t]]_n := A[[t]]/(t^{n+1})$.

Recall from Section 1.2 that a *strongly admissible* k-algebra A is a complete topological k-algebra such that there exists a filtration of ideals

$$\mathfrak{m}_1 \supset \mathfrak{m}_2 \supset \mathfrak{m}_3 \supset \dots$$

which forms a fundamental system of neighborhoods of 0 for the topology of A. We are interested in a criterion for when $A \simeq A'[[s]] \simeq A' \widehat{\otimes}_k k[[s]]$, with A' again strongly admissible. In that case we say that A has a *smooth factor*. The terminology comes from the corresponding geometric statement: A has a smooth factor if and only if there exists a product decomposition $\mathrm{Spf}(A) \simeq \mathrm{Spf}(A') \times \Delta$ of formal k-schemes, where $\Delta = \mathrm{Spf}(k[[s]])$ is often called the *formal disk*. We refer the reader to Section 1.1.4 for more details.

Remark 2.3.1. If $(A, \mathfrak{m})$ is a strongly admissible k-algebra and $A \simeq A'[[s]]$ for some topological ring A', then A' is already strongly admissible.

Let A, C be strongly admissible and $D = (D_i)_i : A \to C$ a higher derivation of order n. We say that D is continuous if every component D_i is a continuous k-linear map. We denote the set of continuous higher derivations of order n by $\mathrm{HS}_k^{n,\mathrm{cont}}(A, C)$. As in Section 2.1, for each such D we denote by γ_D the corresponding map $A \to C[[t]]_n$.

Lemma 2.3.2. *The higher derivation D is continuous if and only if γ_D is, i.e., we have natural isomorphisms*

$$\mathrm{HS}_k^{n,\mathrm{cont}}(A, C) \simeq \mathrm{Hom}^{\mathrm{cont}}_{\mathrm{Alg}_k}(A, C[[t]]_n).$$

Proof. Write $B = C[[t]]_n$ and let $\{\mathfrak{a}_i\}$, $\{\mathfrak{c}_i\}$ be filtrations of ideals defining the respective topologies on A and C. By Proposition 1.2.21, the filtration given by

$$\mathfrak{b}_l = \sum_{i+j=l} t^j \cdot \mathfrak{c}_i[[t]]_n$$

defines the topology on B.

Assume first that there exists a sequence (x_m) in A with $x_m \in \mathfrak{a}_m$ such that $\gamma_D(x_m) \notin \mathfrak{b}_l$ for some $l > 0$. Thus, for each $m \in \mathbb{N}$ there exists a $i_m < l$ such that $D_{i_m}(x_m) \notin \mathfrak{c}_{l-i_m}$. Passing to a suitable subsequence of (x_m), we may assume that $D_i(x_m) \notin \mathfrak{c}_{l-i}$ for all $m \in \mathbb{N}$, thus D_i is not continuous.

Conversely, if there exists a sequence (x_m) in A with $D_i(x_m) \notin \mathfrak{c}_l$ for some fixed i and l, we see that $\gamma_D(x_m) \notin \mathfrak{b}_{l+i}$ for all $m \in \mathbb{N}$. □

Let now $n = \infty$ and $D = (D_i)_{i \geq 0} : A \to C$ a higher derivation. We say that D is *regular* if there exists an element $x \in A$, which is contained in an ideal of definition of A, and such that $D_1(x)$ is invertible in C. Note that x being contained in an ideal of definition is equivalent to x being *topologically nilpotent*, that is, x^n converges to 0 in A as $n \to \infty$.

We start with a quick remark.

Lemma 2.3.3. *If $D = (D_i)_i : A \to C$ is a higher derivation and x contained in an ideal of definition with $D_1(x)$ invertible. Then:*

1. *There exists a higher derivation $D' = (D'_i) : A \to C$ with $D'_1(x) = 1$ and $D'_i(x) = 0$ for $i \geq 0$.*
2. *If $C = A$ and $D_0 = \mathrm{id}_A$ then $x \in A$ is* regular*, that is, not a zero divisor.*

Note that (1) is equivalent to the existence of D' with $\gamma_{D'}(x) = x + t$.

Proof. Let us prove (1) first. By assumption we have that $\gamma_D(x) = \sum_i D_i(x)t^i$ with $D_1(x)$ invertible. Consider the C-linear endomorphism $\varphi : C[[t]] \to C[[t]]$ given by $t \mapsto \gamma_D(x) - D_0(x)$, which is continuous by assumption. Choosing a filtration of ideals $\mathfrak{c}_i$ defining the topology on C and such that $x \in \mathfrak{c}_1$ we get that

$$\mathrm{gr}(\varphi) : \mathrm{gr}(C) \otimes_k \mathrm{gr}(k[[t]]) \to \mathrm{gr}(C) \otimes_k \mathrm{gr}(k[[t]])$$

is an isomorphism and hence so is φ. Composing γ_D with φ^{-1} gives the result.

To see (2), assume $xy = 0$ for some $y \in A$. Then, for each $i \geq 1$ we have $xD_i(y) + D_{i-1}(y) = 0$. In particular, y is divisible by all powers x^n. Since A is separated that implies $y = 0$. □

Before we move on to the main result of this section let us fix some notation first. If $D : A \to C$ is a continuous higher derivation and $\gamma_D : A \to C[[t]]$ the corresponding continuous k-linear map, write $\gamma_t : A[[t]] \to C[[t]]$ for the t-linear extension of γ_D. Then it is easy to verify that γ_t is continuous, and an automorphism for $C = A$. In fact, taking composition in $\mathrm{Aut}_k^{\mathrm{cont}}(A[[t]])$ as the operation allows one to define a group structure on $\mathrm{HS}_k^{\infty,\mathrm{cont}}(A, A)$. We write $\mathrm{HS}_k^{\infty,\mathrm{cont}}(A)$ for the subgroup $\mathrm{HS}_{A/k}^{\infty,\mathrm{cont}}(A, A)$, i.e. those higher derivations $D : A \to A$ whose 0-th component D_0 is the identity on A.

The following theorem is often referred to as the Zariski–Lipman–Nagata criterion (cf. [Mat89, Exercise 30.1]). The statement for Noetherian complete adic A is well-known; an extension to infinite-variate formal power series rings was proven in [CH17] for characteristic 0. Finally, the most general version can be found in [BS19a]. We will give a proof here under the additional assumption that A is strongly admissible.

Theorem 2.3.4 (Zariski–Lipman–Nagata criterion)**.** *Let A be a strongly admissible k-algebra. Then $A \simeq A'[[s]]$ if and only if there exists $D = (D_i)_i \in \mathrm{HS}_k^{\infty,\mathrm{cont}}(A)$ which is regular.*

Proof. Assume first that $A \simeq A'[[s]]$. Then the A'-linear map $A'[[s]] \to A'[[s]][[t]]$ given by $s \mapsto s + t$ gives rise to a regular continuous higher derivation by Lemma 2.3.2.

Let now $D = (D_i)_i : A \to A$ be a continuous higher derivation and $x \in A$ contained in an ideal of definition with $\gamma_D(x) = x + t$. Let $\gamma : A \to A$ be the composition of γ_D with the evaluation $t \mapsto -x$, which is continuous by assumption. Then $\ker(\gamma)$ equals the closure of the ideal $x \cdot A$, since $\gamma(x) = 0$ and

$$\gamma(a) = a + \sum_{i \geq 1} (-x)^i D_i(a) = 0$$

implies that a can be written as a power series in x. Write $A' := \mathrm{Im}(\gamma)$ and consider the A'-linear map $\psi : A'[[s]] \to A$ given by $s \mapsto x$, which is well-defined since x is contained in an ideal of definition. We first claim that ψ is surjective. Indeed, let $\gamma_t \in \mathrm{Aut}_A(A[[t]])$ be the automorphism corresponding to γ_D and let γ_t^{-1} be its inverse. Setting $t = -x$, then for every $a \in A$ we get

$$a = \gamma_{-x}(\gamma_{-x}^{-1}(a)) = \sum_{i \geq 0} \gamma(D_i^*(a))(-x)^i,$$

where $D^* = (D_i^*)_i$ is the higher derivation corresponding to γ_t^{-1}. Hence every $a \in A$ can be written as a power series in x with coefficients in A'. To show that ψ is injective, assume that there is a relation of the form

$$a_d x^d + a_{d+1} x^{d+1} + \ldots = 0, \; a_i \in A'.$$

By (2) of Lemma 2.3.3 we have that x is a regular element of A, so a_d is in the closure of $x \cdot A$. But this is impossible, since

$$\gamma(\gamma(a)) = \gamma(a) + \gamma(\sum_{i \geq 1} (-x)^i D_i(a)) = \gamma(a),$$

i.e. γ is idempotent. □

Note that, if k has characteristic 0, then any (usual) derivation $d \in \mathrm{Der}_k^{\mathrm{cont}}(A)$ gives rise to a higher derivation $D \in \mathrm{HS}_k^{\infty,\mathrm{cont}}(A)$ by setting $D_i := \frac{1}{i!} d^i$. Moreover, D being regular is clearly equivalent to $d(x)$ invertible for some $x \in A$ which is contained in some ideal of definition. In that case we say that d is regular and we obtain the following:

Corollary 2.3.5. *Let A be a strongly admissible k-algebra and assume that k has characteristic 0. Then A has a smooth factor if and only if there exists a regular derivation $d \in \mathrm{Der}_k^{\mathrm{cont}}(A)$.*

2.3.2 Cancellation of smooth factors and the minimal formal model

As a first application of Theorem 2.3.4 we will prove a cancellation result for local quasi-adic rings (see Definition 1.3.1) which contain their residue field. This result is again well-known for Noetherian adic rings. A weaker version of this result whose proof contains the main argument was given in [CH17] as well as in [BNS16], with the full result first appearing in [BS19a].

Theorem 2.3.6. *Let $(A, \mathfrak{m})$, $(A', \mathfrak{m}')$ be two local equicharacteristic quasi-adic rings with $A/\mathfrak{m} = A'/\mathfrak{m}' = k$ and such that A and A' have no smooth factors. Assume there exists an isomorphism*

$$\varphi : A \mathbin{\widehat{\otimes}}_k k[[u_i \mid i \in I]] \simeq A' \mathbin{\widehat{\otimes}}_k k[[v_j \mid j \in J]]$$

of local quasi-adic rings. Then φ induces an isomorphism

$$\varphi' : A \simeq A'.$$

Remark 2.3.7. Note that the isomorphism φ requires the choice of coefficient fields for A and A', see Corollary 1.3.24.

Proof. Let us write $B := A \widehat{\otimes}_k k[[u_i \mid i \in I]$ and $B' := A' \widehat{\otimes}_k k[[v_j \mid j \in J]]$. Denote by $\iota_A : A \to B$ the canonical map, then id_A factors as

$$A \xrightarrow{\iota_A} B \xrightarrow{\pi_A} A,$$

where π_A is the continuous map given by $u_i \mapsto 0$. Denote by $\iota_{A'}$ and $\pi_{A'}$ the corresponding (continuous) maps for A'.

Recall that the continuous Zariski cotangent spaces of A and A' were defined to be $T_A^* := \mathfrak{m}/\widehat{\mathfrak{m}^2}$ and $T_{A'}^* := \mathfrak{m}'/\widehat{(\mathfrak{m}')^2}$ (cf. Definition 1.3.25). Denote similarly by T_B^* and $T_{B'}^*$ the continuous cotangent spaces of B and B'. Then we have

$$T_B^* \simeq T_A^* \oplus \bigoplus_{i \in I} k \cdot \overline{u_i},\ T_{B'}^* \simeq T_{A'}^* \oplus \bigoplus_{j \in J} k \cdot \overline{v_j}.$$

Write $T^*\varphi$ for the continuous cotangent map of φ. We claim that $T^*\varphi$ restricts to an isomorphism $T_A^* \simeq T_{A'}^*$. Indeed, assume that for some $f \in A$ there exists $j \in J$ such that $\varphi(f) = cv_j + g$ with $c \in k^*$ and $\frac{\partial}{\partial v_j} g(0) = 0$. Write D for the higher derivation in $\mathrm{HS}_k^{\infty,\mathrm{cont}}(B')$ given by the A'-linear map $\gamma_D : B' \to B'[[t]]$ defined by $v_e \mapsto v_e$ for $e \neq j$ and $v_j \mapsto v_j + t$. By assumption we have that $D(\varphi(f))$ is invertible. Note that, if k is of characteristic 0, one could alternatively take the derivation $\frac{\partial}{\partial v_j}$. Consider the composition γ'

$$A \xrightarrow{\iota_A} B \xrightarrow{\varphi} B' \xrightarrow{\gamma_D} B'[[t]] \xrightarrow{\widetilde{\varphi^{-1}}} B[[t]] \xrightarrow{\widetilde{\pi_A}} A[[t]],$$

where $\widetilde{\varphi^{-1}}$ and $\widetilde{\pi_A}$ denote the t-linear extensions of φ^{-1} and π_A. Then it is straightforward to check that γ' gives rise to a regular higher derivation $D' \in \mathrm{HS}_k^{\infty,\mathrm{cont}}(A)$. Together with Theorem 2.3.4 this gives a contradiction to the assumption that A has no smooth factors. Thus $T^*\varphi(T_A^*) \subset T_{A'}^*$. Repeating the same argument in the inverse direction we obtain $T^*\varphi^{-1}(T_{A'}^*) \subset T_A^*$ and thus the desired conclusion.

Consider now the map on associated gradeds

$$\mathrm{gr}(\varphi) : \mathrm{gr}(A) \otimes_k k[u_i \mid i \in I] \simeq \mathrm{gr}(A') \otimes_k k[v_j \mid j \in J].$$

As φ is an isomorphism and A, A' are quasi-adic, the map $\mathrm{gr}(\varphi)$ is entirely determined by the cotangent map $T^*\varphi$ (cf. Lemma 1.3.9). Thus $\mathrm{gr}(\varphi)$ induces an isomorphism between $\mathrm{gr}(A)$ and $\mathrm{gr}(A')$. Consider the continuous map φ' given as the composition

$$A \xrightarrow{\iota_A} B \xrightarrow{\varphi} B' \xrightarrow{\pi_{A'}} A'.$$

Then $\mathrm{gr}(\varphi') = \mathrm{gr}(\varphi)\ |_{\mathrm{gr}(A)}$ and by Proposition 1.2.13 we are done. $\square$

For Noetherian adic rings we derive the following (well-known) consequence:

Corollary 2.3.8. *Let $(A, \mathfrak{m})$ be a local equicharacteristic adic Noetherian ring with $A/\mathfrak{m} = k$. Then there exists a local adic Noetherian $(A', \mathfrak{m}')$ and $d \geq 0$ with $A \simeq A' \widehat{\otimes}_k k[[u_1, \ldots, u_d]]$ and A' has no smooth factors. Moreover, the ring A' is unique up to isomorphism.*

Proof. Existence is proven by induction on $\operatorname{edim}(A) < \infty$ and Theorem 2.3.4, while uniqueness follows from Theorem 2.3.6. $\square$

Recall from Definition 1.6.2 that a weak DGK decomposition of a strongly admissible equicharacteristic local ring $(A, \mathfrak{m})$ is an isomorphism

$$A \simeq A' \,\widehat{\otimes}_k\, k[[u_i \mid i \in I]] \tag{2.3a}$$

with A a local (equicharacteristic) Noetherian adic ring. We call (2.3a) a DGK decomposition if A' is isomorphic to the completion of a local ring essentially of finite type over k. By Corollary 2.3.8, if there exists a weak DGK decomposition of A, then there exists one with A' having no smooth factors and unique up to isomorphism with this property.

Lemma 2.3.9. *Let $(A, \mathfrak{m})$ be a local ring essentially of finite type over a field k. Assume that*

$$\varphi : \widehat{A} \simeq A' \,\widehat{\otimes}_k\, k[[u_1, \dots, u_d]].$$

Then $A' \simeq \widehat{B}$, with $\widehat{B}$ the completion of a a local ring $(B, \mathfrak{n})$ essentially of finite type over k.

Proof. Consider $T^*\varphi^{-1}(u_1), \dots, T^*\varphi^{-1}(u_d)$ in $T^*_{\widehat{A}} \simeq T^*_A$ and take liftings $v_1, \dots, v_d \in A$. Set $B := A/(v_1, \dots, v_d)$. Then it is straightforward to verify that the composition

$$A' \longrightarrow A' \,\widehat{\otimes}_k\, k[[u_1, \dots, u_d]] \longrightarrow \widehat{A} \longrightarrow \widehat{B}$$

is an isomorphism. $\square$

Translating the above results into the language of formal schemes leads us to the following definition:

Definition 2.3.10. Let X be a a scheme over a field k and $x \in X(k)$. Assume that there exists a DGK decomposition of x, that is, an isomorphism of formal neighborhoods

$$\widehat{X}_x \simeq \widehat{Y}_y \times \Delta^I,$$

where Y is a scheme of finite type over k, $y \in Y(k)$ and Δ^I is the product of $\Delta = \operatorname{Spf}(k[[t]])$ over some set I. Then we call $\widehat{Y}_y$ a *formal model* for $x \in X$. In this case, by Corollary 2.3.8 and Lemma 2.3.9 there exists a formal model $\widehat{Z}_z$ which has no smooth factors, and which is unique up to isomorphism with this property. We refer to $\widehat{Z}_z$ as the *minimal formal model* for x.

Corollary 2.3.11. *Let X be a scheme over a field k and $x \in X(k)$. Then X is formally smooth at x if and only if* $\operatorname{Spf}(k)$ *is a minimal formal model for x.*

Proof. Note that the completion $\widehat{\mathcal{O}_{X,x}}$ is a local quasi-adic ring. Then we can apply Corollary 1.3.35. $\square$

The minimal formal model should thus be thought of as a formal invariant describing the singularity $x \in X$. The main idea behind this definition is that, even if X itself is infinite-dimensional, the "singular information" of X at a point x might be encoded in a finite-dimensional object. The Drinfeld-Grinberg-Kazhdan theorem (see Theorem 3.4.1) guarantees the existence of formal models for non-degenerate arcs and the study of such formal models will be the subject of Sections 3.4 and 3.5.

2.4 Minimal formal models of algebraic varieties

Although our motivation for introducing the minimal formal model is to study singularities of arc spaces - which are non-Noetherian schemes - it seems instructive to first consider minimal formal models of algebraic varieties. By an *algebraic variety* here we mean an integral separated scheme of finite type over a field k. All of the results we mention here have been studied extensively for (finite-dimensional) analytic spaces, and we believe that they are well-known to experts in the algebraic setting.

We first want to show that the function which associates to each (closed) point the dimension of its minimal formal model is *upper semi-continuous*. Recall that a function $s : X \to \mathbb{N}$ from a topological space X is called upper semi-continuous if, for each $m \in \mathbb{N}$, the set

$$\{q \in X \mid s(q) \geq m\}$$

is closed. This is equivalent to saying that, for each $p \in X$, there exists an open neighborhood U of p such that $s(q) \leq s(p)$ for all $q \in U$. Most invariants used in study of singularities and their resolutions are upper semi-continuous, which can be summarized by the heuristic saying that singularities might get worse by passing to special points.

Assume that k is a field of characteristic 0. Let $X \subset \mathbb{A}^n_k$ be affine, defined by the ideal $\mathfrak{a} \subset k[x_1, \ldots, x_n]$. We define

$$\mathrm{Der}_X := \{d \in \mathrm{Der}(k[x_1, \ldots, x_n]) \mid d(\mathfrak{a}) \subset \mathfrak{a}\}.$$

Elements in Der_X correspond to vector fields on $\mathbb{A}^n_k$ tangent to X. In particular, for $p \in X(k)$ and $d \in \mathrm{Der}_X$ we write $d(p) \in T_pX$ for the tangent vector to X obtained in the natural way.

Proposition 2.4.1. *Let $p \in X(k)$. Then $\widehat{X}_p \simeq \widehat{Y}_y \times \Delta^m$ if and only if there exist m derivations $d_1, \ldots, d_m \in \mathrm{Der}_X$ such that $d_1(p), \ldots, d_m(p) \in T_pX$ are linearly independent over k.*

Proof. Assume first that $d_1, \ldots, d_m \in \mathrm{Der}_X$ are given as above. These induce $\widehat{d}_1, \ldots, \widehat{d}_m \in \mathrm{Der}(\widehat{\mathcal{O}_{X,p}})$ with $\widehat{d}_i(p) = d_i(p)$. Iteratively applying Corollary 2.3.5 gives the desired isomorphism.

Now assume that $\widehat{X}_p \simeq \widehat{Y}_y \times \Delta^m$. Applying Corollary 2.3.5 again in the converse direction guarantees the existence of $\widehat{d}_1, \ldots, \widehat{d}_m \in \mathrm{Der}(\widehat{\mathcal{O}_{X,p}})$ with $\widehat{d}_i(p)$ linearly independent. We may assume that X is defined by $f_1, \ldots, f_r \in k[x_1, \ldots, x_n]$; and that $p = 0$. Lifting the derivations $\widehat{d}_i$ to $\widehat{(\mathbb{A}^n)}_0$ as above we may assume that

$$\widehat{d}_i = \sum_{j=1}^{n} \widehat{d}_{i,j} \partial_{x_j},$$

with $\widehat{d}_{i,j} \in k[[x_1, \ldots, x_m]]$ and $\widehat{d}_i(f_l) \subset (f_1, \ldots, f_r)$. The latter condition is equivalent to

$$\sum_{j=1}^{n} \widehat{d}_{i,j} \frac{\partial f_l}{\partial x_j} = \sum_{j=1}^{r} \widehat{a}_j^{(i,l)} f_j$$

for $i = 1, \dots, m$ and $l = 1, \dots, r$ and with $\widehat{a}_j^{(i,l)} \in k[[x_1, \dots, x_n]]$. Note that this gives a system of linear relations of the elements f_l and their derivatives. Since the completion map $k[x_1, \dots, x_n] \to k[[x_1, \dots, x_n]]$ is flat, a standard argument using the relational criterion for flatness yields $d_1, \dots, d_m \in \mathrm{Der}_X$ with $d_1(0), \dots, d_m(0)$ linearly independent. □

Theorem 2.4.2. *Let X be an algebraic variety over an algebraically closed field k of characteristic 0. Consider the function $s : X(k) \to \mathbb{N}_0$ defined by*

$$p \in X(k) \mapsto \dim(\widehat{Y}_y),$$

where $\widehat{Y}_y$ is the minimal formal model of $\widehat{X}_p$. Then s is upper semi-continuous.

Proof. Let $n = \dim(X)$ and $p \in X(k)$ with $s(p) = n - m$. It is enough to show that there exists an open neighborhood U of p such that, for all $q \in U(k)$, we have $s(q) \leq n - m$. Thus we may assume that $X \subset \mathbb{A}^N$ is affine. Since $\widehat{X}_x \simeq \widehat{Y}_y \times \Delta^m$, by Proposition 2.4.1, there exist derivations $d_1, \dots, d_m \in \mathrm{Der}_X$ with $d_1(p), \dots, d_m(p) \in T_pX$ linearly independent. Writing

$$d_i = \sum_{j=1}^{N} d_{i,j} \partial_{x_j},$$

with $d_{i,j} \in k[x_1, \dots, x_N]$, we may assume that $f := \det((d_{i,j})_{1 \leq i,j \leq m}) \neq 0$. The condition $f \neq 0$ defines an open subset U of X. By applying Proposition 2.4.1 in the converse direction, we see that for every $q \in U(k)$, we have $\widehat{X}_q \simeq \widehat{Y}_y \times \mathbb{A}^m$. Since $\dim_q(X) = \dim_p(X) = n$ we see that $s(q) \leq n - m$. □

2.4.1 The isosingular loci of an algebraic variety

We have seen previously that the minimal formal model provides a meaningful invariant for singularities of algebraic varieties. In [Eph78], the author gave a geometric construction for the corresponding notion for *analytic spaces* as follows: for X an analytic space and $p \in X$ let $\mathrm{Iso}(X, p)$ denote the subset of all $q \in X$ such that the analytic germs (X, p) and (X, q) are isomorphic; we call $\mathrm{Iso}(X, p)$ the *isosingular locus* of X at p. Then we have:

Theorem 2.4.3 ([Eph78, Theorem 0.2])**.** *Let X be a (reduced) analytic space and $p \in X$. Then the set $\mathrm{Iso}(X, p)$ is locally closed and smooth as a reduced analytic subspace. Moreover, for each $q \in \mathrm{Iso}(X, p)$, there exists an open neighborhood U of q and an analytic space Y such that $U \simeq Y \times (U \cap \mathrm{Iso}(X, p))$.*

For a slightly different proof see also [HM90, Theorem 3] and [HM89].

As was remarked in [Eph78, Observation 2.5], the decomposition in Theorem 2.4.3 is maximal, in the sense that Y is not locally isomorphic to $Y' \times (\mathbb{C}, 0)$. Thus the germ of Y is the minimal *analytic* model of $p \in X$, transversal to $\mathrm{Iso}(X, p)$.

Our aim in this section is to consider a notion of isosingular loci for schemes of finite type over a field k and adapt the arguments of [Eph78] and [HM90] to the algebraic setting. As we do not have the analytic topology at our disposal, we will instead consider isomorphisms of the corresponding formal neighborhoods (which, as we will see later, is equivalent to working with the *étale topology*). Here is the main definition:

Definition 2.4.4. Let X be a scheme of finite type over a field k and let $p \in X(k)$. The *isosingular locus* of X at p is the set

$$\mathrm{Iso}(X,p) := \{q \in X(k) \mid \widehat{X}_q \simeq \widehat{X}_p\},$$

where $\widehat{X}_p$ and $\widehat{X}_q$ denote the respective formal neighborhoods of p and q.

Example 2.4.5. Let k be an algebraically closed field and X the *Whitney umbrella* over k defined by $f = x^2 + ty^2$ in $\mathbb{A}^3_k$. Its singular locus is given by $Z = \{x = y = 0\}$. For $p = (x, y, t)$ we can distinguish between three cases:

1. If $p \in X \setminus Z$, then $\mathrm{Iso}(X, p) = X \setminus Z$ is the smooth locus.
2. If $p \in Z$ and $t \neq 0$, then $\widehat{X}_p \simeq \widehat{Y}_y \times \widehat{\mathbb{A}^1}_0$, where $Y = \mathrm{Spec}(k[x, y]/(xy))$ is the union of two lines. Thus $\mathrm{Iso}(X, p) = Z \setminus \{0\}$.
3. If $p = 0$, then $\widehat{X}_p$ has no smooth factors. Then $\mathrm{Iso}(X, 0) = \{0\}$.

Thus in this case we see that the isosingularity loci are all locally closed as subsets of X. However, for that to hold, note that the assumption that k is algebraically closed is essential. For example, for $p = (0, 0, t) \in X(k)$, $t \neq 0$, the formal neighborhood $\widehat{X}_p$ has two components if and only $\sqrt{t} \in k$. In that case the isosingularity locus $\mathrm{Iso}(X, p)$ might not even be constructible as a subset of X.

Our goal is thus to prove an algebraicity result for $\mathrm{Iso}(X, p)$ over algebraically closed fields. To that avail, we will follow [Eph78] and [HM90] and describe formal equivalence by the action of the *contact group*. We first prove that $\mathrm{Iso}(X, p)$ is locally closed and smooth when considered as a reduced subscheme of X over arbitrary algebraically closed fields. To obtain the full analogue of Theorem 2.4.3 as Theorem 2.4.16 we will make use of the Zariski-Lipman-Nagata criterion and thus restrict to characteristic 0. We do not know of an example where the decomposition fails to exist in positive characteristics.

Let us first introduce some necessary terminology. Recall that a *group scheme* is a k-scheme G which is a group object in the category of k-schemes, that is, it comes equipped with morphisms $m : G \times G \to G$, $i : G \to G$ and $e : \mathrm{Spec}(k) \to G$ corresponding to multiplication, inversion and the neutral element and satisfying the usual group axioms. In particular, for each k-algebra R the map $m(R)$ defines a group structure on the set $G(R)$. We say that a group scheme G is an *algebraic group* if it is of finite type over k. A *group action* of G on some scheme X is a morphism $a : G \times X \to X$ such that, for each k-scheme S, the map $G(S) \times X(S) \to X(S)$ satisfies the usual properties of a group action with respect to $G(S)$. If $x \in X(k)$ is a k-rational point, then we may consider $x \in X(S)$ in the natural way. The *orbit map* is the morphism $a_x : G \to X$ given by $g \in G(S) \mapsto g \cdot x \in X(S)$. It is well-known that the image of a_x is locally closed in X. Moreover, since we assumed the base field k to be algebraically closed, we have that the k-rational points of the $\mathrm{Im}(a_x)$ are just the (set-theoretic) orbit $G(k) \cdot x$.

Let us introduce some notation. We assume that we are working over an algebraically closed field k; in particular, R is assumed to be a k-algebra and all schemes constructed are schemes over k.

- By $\mathcal{O}_n$ we denote the (infinite-dimensional) scheme whose R-points are formal power series in $R[[x_1, \ldots, x_n]]$.

- For $\beta > 0$ the scheme $\mathcal{O}_{n,\beta}$ is the (finite-dimensional) scheme whose R-points are truncated power series in

$$R[[x_1, \ldots, x_n]]/(x_1, \ldots, x_n)^{\beta+1}.$$

  Note that the natural quotient map induces a morphism $\mathcal{O}_n \to \mathcal{O}_{n,\beta}$.

- For $r > 0$ we consider the group scheme $\mathrm{GL}_r(\mathcal{O}_n)$ acting on $\mathcal{O}_n^r$ and the scheme $\mathrm{Aut}(\mathcal{O}_n)$ whose R-points are $\varphi = (\varphi_1, \ldots, \varphi_n) \in R[[x_1, \ldots, x_n]]^n$ satisfying $\varphi(0) = 0$ and $\det((\frac{\partial \varphi_i}{\partial x_j})_{i,j}) \in R^*$. By the formal inverse function theorem (see Lemma 1.2.24) $\mathrm{Aut}(\mathcal{O}_n)$ is a group scheme with operation given by substitution, which acts on $\mathrm{GL}_r(\mathcal{O}_n)$ in the natural way. Consider the diagonal action of $\mathrm{Aut}(\mathcal{O}_n)$ on $\mathcal{O}_n^r$; then the semidirect product $\mathcal{K} := \mathrm{GL}_r(\mathcal{O}_n) \rtimes \mathrm{Aut}(\mathcal{O}_n)$ acts on $\mathcal{O}_n^r$. We call $\mathcal{K}$ the *contact group.*

- Arguing as before for $\mathcal{O}_{n,\beta}$ instead of $\mathcal{O}_n$ we obtain as $\mathcal{K}_\beta := \mathrm{GL}_r(\mathcal{O}_{n,\beta}) \rtimes \mathrm{Aut}(\mathcal{O}_{n,\beta})$ the *truncated contact group.* Truncation induces a morphism of group schemes $\mathcal{K} \to \mathcal{K}_\beta$ which satisfies the following commutative diagram

$$\begin{tikzcd}
\mathcal{K} \times \mathcal{O}_n^r \arrow[r] \arrow[d] & \mathcal{O}_n^r \arrow[d] \\
\mathcal{K}_\beta \times \mathcal{O}_{n,\beta}^r \arrow[r] & \mathcal{O}_{n,\beta}^r .
\end{tikzcd}$$

Let us spell out the action of $\mathcal{K}$ on $\mathcal{O}_n^r$. We write $x = (x_1, \ldots, x_n)$. Let $f = (f_1, \ldots, f_r) \in R[[x]]^r$, $M \in (R[[x]])^{r \times r}$ and $\varphi = (\varphi_1, \ldots, \varphi_n) \in \mathrm{Aut}(R[[x]])$. Then

$$(M, \varphi) \cdot f(x) = M \cdot f(\varphi(x)).$$

Now let $X \subset \mathbb{A}^n$ be an affine scheme over k defined by the system of equations $f = (f_1, \ldots, f_r)$. For each R, the R-points of X can be identified with $a = (a_1, \ldots, a_n) \in R^n$ satisfying $f(a) = 0$. Then we consider the morphism $\gamma : X \to \mathcal{O}_n^r$ given by

$$(a_1, \ldots, a_n) \mapsto (f_1(x_1 + a_1, \ldots, x_n + a_n), \ldots, f_r(x_1 + a_1, \ldots, x_n + a_n)),$$

which we abbreviate by $a \mapsto f(x + a)$. The main idea is that the formal neighborhoods of two k-points p and q are isomorphic if and only if the respective Taylor expansions $\gamma(p)$ and $\gamma(q)$ of f lie in the same orbit of the contact group. More precisely, let $p \in X(k)$, and consider the orbit map $\gamma_p : \mathcal{K} \to \mathcal{O}_n^r$ given by $g \mapsto g \cdot \gamma(p)$. Then we have:

Lemma 2.4.6. *Let X, γ be as before and $p, q \in X(k)$. Then $\widehat{X}_p \simeq \widehat{X}_q$ if and only if $\gamma(q) \in \mathrm{Im}(\gamma_p)$.*

Proof. We may assume that $p = 0$ and $q = (a_1, \ldots, a_n)$. Then

$$\widehat{\mathcal{O}_{X,q}} \simeq k[[x_1, \ldots, x_n]]/(f(x + a)).$$

The existence of an isomorphism $\widehat{\mathcal{O}_{X,0}} \simeq \widehat{\mathcal{O}_{X,q}}$ is equivalent to the existence of an isomorphism $\varphi \in \mathrm{Aut}_k(k[[x_1, \ldots, x_n]])$ such that the following equality of ideals holds:

$$(\varphi(f(x))) = (f(x+a)).$$

To get the statement we make use of the following trick of Mather (see [Mather]):

Lemma 2.4.7. *Let $A, B \in k^{n\times n}$. Then there exists $C \in k^{n\times n}$ such that $C(1 - AB) + B$ is invertible.*

Proof. Let $r = \mathrm{rk}(B)$ and choose a basis $\{e_i\}_i$ for k^n such that $Be_1, \ldots, Be_r$ are linearly independent and $Be_i = 0$ for $i > r$. Let $e'_{r+1}, \ldots, e'_n$ be such that Be_i, e'_j form a basis. Then C is the matrix representing the linear map given by $e_i \mapsto 0$ for $i \leq r$ and $e_i \mapsto e'_i$ for $i > r$. □

Write $R = k[[x_1, \ldots, x_n]]$ and $\mathfrak{m} = (x_1, \ldots, x_n)$. The above equality of ideals implies the existence of $A, B \in R^{r\times r}$ with $A \cdot f(\varphi(x)) = f(x + a)$ and $B \cdot f(x + a) = f(\varphi(x))$. Then, by Lemma 2.4.7, there exists $C \in k^{r\times r}$ with $D = C(1 - AB) + B \in R^{r\times r}$ invertible modulo $\mathfrak{m}$, which implies that D is invertible. Clearly, $D \cdot f(x + a) = f(\varphi(x))$.

□

Unfortunately, the group scheme $\mathcal{K}$ is not of finite type, and neither is the scheme $\mathcal{O}_n^r$ - in particular, we cannot directly apply the established results to see that $\mathrm{Im}(\gamma_p)$ is locally closed! Our way around this is to make use of the following variant of the famous Artin approximation theorem, which is commonly referred to as *Universal Strong Artin Approximation.*

Theorem 2.4.8 ([Art69, Theorem 6.1])**.** *Let k be a field. For each tuple (n, m, d, α) of non-negative integers there exists a $\beta \geq 0$ satisfying the following property: let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$ be two sets of variables and $f = (f_1, \ldots, f_r) \in k[x, y]^r$ with $\deg(f_i) \leq d$. Assume there exist polynomials $\bar{y} = (\bar{y}_1, \ldots, \bar{y}_m) \in k[x]^m$ with*

$$f(x, \bar{y}) \equiv 0 \mod (x)^\beta.$$

Then there exist algebraic power series $y(x) = (y_1(x), \ldots, y_m(x))$ with $f(x, y(x)) = 0$ and

$$\bar{y} \equiv y(x) \mod (x)^\alpha.$$

Going back, let $\beta > 0$ and write $\gamma_\beta : X \to \mathcal{O}_{n,\beta}^r$ for the composition of γ with the natural truncation morphism $\mathcal{O}_n^r \to \mathcal{O}_{n,\beta}^r$. For $p \in X(k)$ we consider the corresponding orbit map $(\gamma_\beta)_p : \mathcal{K}_n \to \mathcal{O}_{n,\beta}^r$.

Corollary 2.4.9. *Let X be as before. Then there exists a $\beta > 0$ such that, for $g \in X(k)$ we have $\gamma(q) \in \mathrm{Im}(\gamma_p)$ if and only if $\gamma_\beta(q) \in \mathrm{Im}((\gamma_\beta)_p)$.*

Proof. Let q be given by $a = (a_1, \ldots, a_n) \in k^n$. Consider the system of equations in variables $M = (M_{ij})$ and $y = (y_i)$ over $k[x_1, \ldots, x_n]$ given by

$$M \cdot f(y_i) - f(x + a) = 0.$$

The condition $\gamma_\beta(q) \in \mathrm{Im}((\gamma_\beta)_p)$ is equivalent to the existence of $(\bar{M}(x), \bar{y}(x))$ such that $\det(\bar{M}(0))$, $\det(\frac{\partial \bar{y}_i}{\partial x_j}(0)) \in k^*$. Take $\alpha >> 0$. By Theorem 2.4.8 there a $\beta > 0$ such that, for each solution $(\bar{M}(x), \bar{y}(x))$ of this system module $(x)^\beta$, there exists a solution $(M(x), y(x))$ with $\bar{M}(x) - M(x), \bar{y}(x) - y(x) \in (x)^\alpha$. Thus, in particular $\det(M(0))$, $\det(\frac{\partial y_i}{\partial x_j}(0)) \in k^*$, which in turn implies that $\gamma(q) \in \mathrm{Im}(\gamma_p)$. The other direction is trivial. $\square$

Thus we obtain the following result for schemes of finite type:

Theorem 2.4.10. *Let X be a scheme of finite type over an algebraically closed field k and let $p \in X(k)$. Then $\mathrm{Iso}(X, p)$ is locally closed as a subset of $X(k)$.*

Proof. Let $\{U_i\}$ be an open affine covering of X, then it is enough to show that $U_i \cap \mathrm{Iso}(X, p)$ is locally closed for each $i \in I$. Thus we can assume that $X \subset \mathbb{A}^n$ is given by $f_1, \ldots, f_r \in k[x_1, \ldots, x_n]$ and $p = 0 \in X$. Lemma 2.4.6 together with Corollary 2.4.9 show that, for any $q \in X(k)$, we have $\widehat{X}_q \simeq \widehat{X}_p$ if and only if $\gamma_\beta(q) \in \mathrm{Im}((\gamma_\beta)_p)$. As the image of the orbit map $(\gamma_\beta)_p$ is locally closed, so is $\mathrm{Iso}(X, p) = \gamma_\beta^{-1}(\mathrm{Im}((\gamma_\beta)_p)(k))$. $\square$

Theorem 2.4.10 leads us the make the following definition:

Definition 2.4.11. Let X be a scheme of finite type over an algebraically closed field k. For each $p \in X(k)$ we define $X^{(p)}$ to be the unique reduced subscheme of X whose k-points equal $\mathrm{Iso}(X, p)$ and call it the *isosingularity scheme* associated to p.

The next goal is to show that $X^{(p)}$ is smooth. The strategy is to use generic smoothness to find a $q \in \mathrm{Iso}(X, p)$ such that $X^{(p)}$ is smooth at q and then extend the isomorphism $\widehat{X}_p \simeq \widehat{X}_q$ *étale-locally.* To that end, recall that an *étale neighborhood* (U, x') of $x \in X(k)$ is an étale morphism $u : U \to X$ and $x' \in U(k)$ with $u(x') = x$. Artin's approximation results then imply the following corollary:

Lemma 2.4.12 ([Art69, Corollary 2.5])**.** *Let $p \in X(k)$ and $q \in \mathrm{Iso}(X, p)$, that is, $\widehat{X}_p \simeq \widehat{X}_q$. Then there exists a common étale neighborhood (U, p') of p and q, that is, a diagram of étale morphisms*

$$\begin{tikzcd} & U \arrow[dl, "u"'] \arrow[dr, "u'"] & \\ X & & X \end{tikzcd}$$

and $p' \in U(k)$ with $u(p') = p$ and $u'(p') = q$.

Lemma 2.4.13. *Let $f : U \to X$ be étale and $p' \in U(k)$ with $p = f(p') \in X(k)$. Then $f^{-1}(\mathrm{Iso}(X, p)) = \mathrm{Iso}(U, p')$ and the restriction $U^{(p')} \to X^{(p)}$ is étale.*

Proof. The first assertion follows from the fact that, for $q' \in U(k)$ and $q = f(q')$ the morphism f induces an isomorphism on completions $\widehat{U}_{q'} \simeq \widehat{X}_q$. To see the second claim, consider the fiber diagram

$$\begin{tikzcd} U \times_X X^{(p)} \arrow[r] \arrow[d] & X^{(p)} \arrow[d] \\ U \arrow[r, "f"'] & X. \end{tikzcd}$$

As a base change of f the morphism $U \times_X X^{(p)} \to X^{(p)}$ is étale again and in particular, since $X^{(p)}$ is reduced, so is $U \times_X X^{(p)}$. Thus $U^{(p')} \simeq U \times_X X^{(p)}$ and we are done. $\square$

Proposition 2.4.14. *Let X be a scheme of finite type over an algebraically closed field k and let $p \in X(k)$. Then $X^{(p)}$ is smooth over k.*

Proof. By definition $X^{(p)}$ is (geometrically) reduced and thus the subset of k-smooth points of $X^{(p)}$ is dense open (see [SP, Tag 056V]). Thus there exists $q \in X^{(p)}(k) = \mathrm{Iso}(X, p)$ smooth over k. By Lemma 2.4.12 there exists a common étale neighborhood (U, p') of p and q. Thus, by Lemma 2.4.13, we have $\widehat{(X^{(p)})}_p \simeq \widehat{(U^{(p')})}_{p'} \simeq \widehat{(X^{(q)})}_q$. Clearly $X^{(q)} \simeq X^{(p)}$, and thus $\widehat{(X^{(p)})}_p$ is formally smooth over k, which in turn implies that $X^{(p)}$ is smooth at p. $\square$

Let us also make the following observation:

Lemma 2.4.15. *Let X be a scheme of finite type over an algebraically closed field k and let $p \in X(k)$. Assume that $\widehat{X}_p \simeq \widehat{Y}_y \times \Delta^m$. Then $\dim_p X^{(p)} \geq m$.*

Proof. By Lemma 2.4.12 we can find a common étale neighborhood U for $p \in X$ and $y' = (y, 0) \in Y \times \mathbb{A}^m$. Clearly $\mathrm{Iso}(Y \times \mathbb{A}^m, y') \supset \{y\} \times \mathbb{A}^m$ and thus we are done using Lemma 2.4.13. $\square$

Now, in order to prove the existence of a decomposition as in Theorem 2.4.3, we will make use of Theorem 2.3.4 and thus have to assume that our base field k is of characteristic 0. As the proof is essentially the same as in [HM90] we will only give a sketch here, elaborating on the modifications when carrying over arguments used in the analytic setting.

Let $X \subset \mathbb{A}^n$ thus be an affine algebraic variety over an algebraically closed field k of characteristic 0 and let $p \in X(k)$. As before, let X be given by $f_1, \dots, f_r$ and for $\beta > 0$ consider the corresponding map $\gamma_\beta : X \to \mathcal{O}^r_{n,\beta}$. As k is of characteristic 0, the orbit map $(\gamma_\beta)_p : \mathcal{K}_\beta \to \mathcal{O}^r_{n,\beta}$ is smooth onto its image, which we will denote Z. Thus, the induced map of formal neighborhoods

$$\widehat{(\mathcal{K}_\beta)}_1 \to \widehat{Z}_{\gamma_\beta(p)},$$

with 1 denoting the unit in $\mathcal{K}_\beta$, allows for a section $\widehat{Z}_{\gamma(p)} \to \widehat{(\mathcal{K}_\beta)}_1$. Writing $\widehat{\Phi}_\beta : \widehat{(X^{(p)})}_p \to \widehat{(\mathcal{K}_\beta)}_1$ for the composition with the morphism $\widehat{(X^{(p)})}_p \to \widehat{Z}_{\gamma_\beta(p)}$ we obtain a factorization

$$\widehat{(X^{(p)})}_p \xrightarrow{\widehat{\Phi}_\beta} \widehat{(\mathcal{K}_\beta)}_1 \xrightarrow{\widehat{(\gamma_\beta)_p}} \widehat{Z}_{\gamma_\beta(p)} .$$

Now, with a similar argument making use of Strong Artin approximation as before (see [HM90; HM89] for details), we get a factorization of $\widehat{\gamma}$ as

$$\widehat{(X^{(p)})}_p \xrightarrow{\widehat{\Phi}} \widehat{\mathcal{K}}_1 \xrightarrow{\widehat{\gamma_p}} \widehat{Z}_{\gamma(p)} .$$

For convenience's sake let assume $p = 0$ from now on. The key idea is to use the above factorization to compute the tangent map of $\widehat{\gamma}$, which we will do by considering the appropriate functor of points. Let $(A, \mathfrak{m})$ be a test ring with

$A/\mathfrak{m} = k$ and $\mathfrak{m}^2 = 0$. Working with coordinates, for each $a = (a_1, \dots, a_n) \in \widehat{(X^{(0)})}_0(A)$ we write $\widehat{\Phi}(a) = (u(a), x + \varphi(a))$, with $u \in \mathrm{GL}_r(A[[x,a]])$ and $\varphi \in k[[x,a]]^n$ satisfying $u(0) = 1$ and $\varphi(0) = 0$, $\mathrm{ord}_x(\varphi) = 1$. Then the tangent map of $\widehat{\gamma}_p \circ \widehat{\Phi}$ is given by

$$a \mapsto u(a) \cdot f(x + \varphi(a)).$$

Using Taylor expansion at $a = 0$, we have

$$u(a) \cdot f(x + \varphi(a)) = f(x) + a^T \cdot \left(\frac{\partial u}{\partial x}(0) \cdot f + \frac{\partial f}{\partial x} \cdot \frac{\partial \varphi}{\partial x}(0) \right).$$

On the other hand, the tangent map of $\widehat{\gamma}$ is simply

$$a \mapsto f(x + a),$$

which, after using Taylor expansion again, yields

$$a^T \cdot \left(\frac{\partial u}{\partial x}(0) \cdot f + \frac{\partial f}{\partial x} \cdot \frac{\partial \varphi}{\partial x}(0) \right) = a^T \cdot \frac{\partial f}{\partial x}.$$

Note that, by assumption, we have that $\frac{\partial \varphi}{\partial x}(0)$ is of order 1. Thus any choice of tangent vector $a \in T_0 X^{(0)}$ gives a derivation $d \in \mathrm{Der}_k(k[[x]])$ satisfying $d(f) \in (f)$ and $d(0) = a$. Since $X^{(0)}$ is smooth of dimension $m = \dim T_0 X^{(0)}$, Proposition 2.4.1 gives an isomorphism

$$\widehat{X}_0 \simeq \widehat{(X^{(0)})}_0 \times \widehat{Y}_y.$$

By Lemma 2.4.15 we see that $\widehat{Y}_y$ has no smooth factors. To summarize, we have proven the analogue of Theorem 2.4.3 in the algebraic case:

Theorem 2.4.16. *Let X be a scheme of finite type over an algebraically closed field of characteristic 0. Let $p \in X(k)$ and let $X^{(p)}$ be the associated isosingularity scheme. Then there exists an isomorphism*

$$\widehat{X}_p \simeq \widehat{Y}_y \times \widehat{(X^{(p)})}_p,$$

compatible with the closed immersion $X^{(p)} \to X$ and such that $\widehat{Y}_y$ has no smooth factors, i.e. is a minimal formal model for p.

2.5 Embedding codimension

Let X be a variety over k and $p \in X(K)$. In Section 2.3 we considered the dimension of the minimal formal model of p as a (rough) invariant measuring the "size" of the singularity at p. A different (and in some sense even coarser) invariant is the *embedding codimension* (or *regularity defect*) of X at p, which is just the difference

$$\mathrm{edim}(\mathcal{O}_{X,p}) - \dim_p(X).$$

Clearly this quantity is 0 if and only if $p \in X$ is regular. Of course, if we instead consider an infinite-dimensional scheme X, the above definition is no longer meaningful. In this section we will thus introduce two possible ways of generalizing the embedding codimension to the non-Noetherian case, one using the associated graded ring and the other the respective formal completions. In Theorem 2.5.19 we will prove an inequality relating these two notions; we do not know if they differ in general.

2.5.1 Embedding codimension

Throughout this section we will denote by $(A, \mathfrak{m}, k)$ a local quasi-preadic ring, see Section 1.3. Recall that this means that A is endowed with a filtration

$$\mathfrak{m}_1 \supset \mathfrak{m}_2 \supset \mathfrak{m}_3 \supset \dots$$

such that $\mathfrak{m}_1 = \mathfrak{m}$ and $\mathfrak{m}_n/\mathfrak{m}_{n+1} \simeq \mathfrak{m}^n/\mathfrak{m}_{n+1}$. By Lemma 1.3.9 this is equivalent to the natural map

$$\gamma\colon \operatorname{Sym}_k(\mathfrak{m}/\mathfrak{m}_2) \to \operatorname{gr}(A).$$

being surjective.

Definition 2.5.1. The *embedding codimension* of $(A, \mathfrak{m}, k)$ is defined to be

$$\operatorname{ecodim}(A) := \operatorname{ht}(\ker(\gamma)).$$

Proposition 2.5.2. *For any local quasi-preadic ring* $(A, \mathfrak{m}, k)$*, we have*

$$\operatorname{edim}(A) = \dim(\operatorname{gr}(A)) + \operatorname{ecodim}(A).$$

In particular, if A is Noetherian preadic then $\operatorname{edim}(A) = \dim(A) + \operatorname{ecodim}(A)$.

Proof. This follows from the fact that for every polynomial ring $P = k[x_i \mid i \in I]$ and every ideal $\mathfrak{a} \subset P$, we have $\dim(P) = \dim(P/\mathfrak{a}) + \operatorname{ht}(\mathfrak{a})$ (cf. Remark 1.6.11). For the last assertion we use that $\dim(\operatorname{gr}(A)) = \dim(A)$ if A is Noetherian preadic. $\square$

Remark 2.5.3. The formula in Proposition 2.5.2 is still valid, and informative, when some of the quantities involved are infinite.

Remark 2.5.4. Higher rank valuation rings provide examples of finite dimensional non-Noetherian rings whose embedding dimension is smaller than their dimension. In particular, the second equation in Proposition 2.5.2 does not hold for those rings.

Remark 2.5.5. The embedding codimension of a local ring was studied in the Noetherian (preadic) setting in [Lec64] under the name of *regularity defect*. One of the properties proved under some more restrictive hypotheses there is that if A is a local G-ring then $\operatorname{ecodim}(A_{\mathfrak{p}}) \le \operatorname{ecodim}(A)$ for every prime ideal $\mathfrak{p} \subset A$, see [Vin]. It would be interesting to find suitable conditions for the same property to hold in the non-Noetherian setting.

Remark 2.5.6. One should think of the embedding codimension as the codimension of the tangent cone $C = \operatorname{Spec}(\operatorname{gr}(A))$ to $X = \operatorname{Spec}(A)$ inside the tangent space at the closed point of X. Note that only the components of C of highest dimension are counted. Observe also that for irreducible X, C might not be irreducible - however, in this case, all (non-embedded) components of C are of the same dimension.

We now come to the main result of this section, which gives bounds for the embedding codimension of A from maps into A.

Proposition 2.5.7. *Let $\varphi\colon (B, \mathfrak{n}, k_0) \to (A, \mathfrak{m}, k)$ be a homomorphism of local quasi-preadic rings, and assume that $(B, \mathfrak{n})$ has finite embedding dimension. Let $D\varphi\colon \mathfrak{n}/\overline{\mathfrak{n}^2} \otimes_{k_0} k \to \mathfrak{m}/\overline{\mathfrak{m}^2}$ be the continuous cotangent map of φ. Then*

$$\operatorname{ecodim}(A) \ge \operatorname{rank}(D\varphi) - \dim(\operatorname{gr}(B)).$$

In particular, if B is Noetherian preadic then $\operatorname{ecodim}(A) \ge \operatorname{rank}(D\varphi) - \dim(B)$.

Remark 2.5.8. A stronger form of Proposition 2.5.7 is obtained by replacing $\dim(\mathrm{gr}(B))$ with $\dim(\mathrm{gr}(B/\ker(\mathrm{gr}(\varphi))))$ in the displayed formula. Note, in fact, that this sharper form of the proposition follows from the special case of the proposition where φ is assumed to be injective.

Remark 2.5.9. Consider the special case where φ is a homomorphism of local preadic k-algebras with residue fields k (that is, such that the natural maps $k \to B/\mathfrak{n}$ and $k \to A/\mathfrak{m}$ are isomorphisms) and with B essentially of finite type. The geometric interpretation is the following. Let $f\colon X \to Y$ be a morphism of schemes over k, with Y of finite type over k, and let $p \in X(k)$ and $q = f(p) \in Y(k)$. Denote by $T_p f\colon T_p X \to T_q Y$ the map induced on Zariski tangent spaces. Then the proposition gives

$$\mathrm{ecodim}(\mathcal{O}_{X,p}) \geq \dim(\mathrm{Im}(T_p f)) - \dim_q(\mathrm{Im}(f))$$

where $\mathrm{Im}(f) \subset Y$ is the scheme-theoretic image of f. Note in particular that if X is Noetherian then this formula reduces to the intuitive statement that

$$\dim(T_p X) - \dim_p(X) \geq \dim(\mathrm{Im}(T_p f)) - \dim_q(\mathrm{Im}(f)).$$

Another special case is when f is a submersion onto Y, in which case the formula reduces to the inequality $\mathrm{ecodim}(\mathcal{O}_{X,p}) \geq \mathrm{ecodim}(\mathcal{O}_{Y,p})$.

Proof of Proposition 2.5.7. We have the commutative diagram

$$\begin{tikzcd}
\mathrm{Sym}_{k_0}(\mathfrak{n}/\overline{\mathfrak{n}^2}) \arrow[r, "\pi"] \arrow[d, two heads] & \mathrm{Sym}_k(\mathrm{Im}(D\varphi)) \arrow[r, hook, "\iota"] \arrow[d, "\sigma"] & \mathrm{Sym}_k(\mathfrak{m}/\overline{\mathfrak{m}^2}) \arrow[d, two heads, "\gamma"] \\
\mathrm{gr}(B) \arrow[r] \arrow[rr, bend right, "\mathrm{gr}(\varphi)"'] & \mathrm{Im}(\mathrm{gr}(\varphi)) \otimes_{k_0} k \arrow[r, hook, "\psi"] & \mathrm{gr}(A)
\end{tikzcd}$$

The existence of σ follows from the fact that $\mathrm{Im}(\pi) \otimes_{k_0} k = \mathrm{Sym}_k(\mathrm{Im}(D\varphi))$. The map ι is a linear extension of polynomial rings, and hence is faithfully flat. Since $\iota^{-1}(\ker(\gamma)) = \ker(\sigma)$ we see that

$$\mathrm{ht}(\ker(\gamma)) \geq \mathrm{ht}(\ker(\sigma))$$

by the going-down theorem. On the other hand,

$$\mathrm{ht}(\ker(\sigma)) = \mathrm{rank}(D\varphi) - \dim(\mathrm{Im}(\sigma)).$$

Since the inclusion $\mathrm{Im}(\sigma) \subset \mathrm{Im}(\mathrm{gr}(\varphi)) \otimes_{k_0} k$ is an inclusion of Noetherian local rings with the same residue field, and $\mathrm{Im}(\mathrm{gr}(\varphi))$ is a quotient of $\mathrm{gr}(B)$, we have

$$\dim(\mathrm{Im}(\sigma)) \leq \dim(\mathrm{gr}(B)).$$

Combining the above formulas, we get

$$\mathrm{ht}(\ker(\gamma)) \geq \mathrm{rank}(D\varphi) - \dim(\mathrm{gr}(B)).$$

To conclude, notice that $\dim(\mathrm{gr}(B)) = \dim(B)$ if B is Noetherian preadic. $\square$

The following result shows that the embedding codimension of A is invariant under change of the base field, provided the residue field is already contained in A.

Proposition 2.5.10. *Let $(A, \mathfrak{m}, k)$ be a local quasi-preadic k-algebra such that the natural map $k \to A/\mathfrak{m}$ is an isomorphism, and let $k \subset k'$ be a field extension. Denoting $A' := A \otimes_k k'$, we have*

$$\operatorname{ecodim}(A') = \operatorname{ecodim}(A).$$

Proof. First, observe that A' is a local k'-algebra with maximal ideal $\mathfrak{m}' = \mathfrak{m} \otimes_k k'$. Moreover $\mathfrak{m}'_n := \overline{(\mathfrak{m}')^n} = \overline{\mathfrak{m}^n} \otimes_k k'$. We have $\operatorname{ecodim}(A) = \operatorname{ht}(\ker(\gamma))$, where

$$\gamma \colon \operatorname{Sym}_k(\mathfrak{m}/\overline{\mathfrak{m}^2}) \to \operatorname{gr}(A)$$

is defined as before. Since for every n we have $(\mathfrak{m}')_n/(\mathfrak{m}')_{n+1} = \mathfrak{m}_n/\mathfrak{n}_{n+1} \otimes_k k'$, we see that γ induces, by base change, the analogous map

$$\gamma' \colon \operatorname{Sym}_k(\mathfrak{m}'/\overline{(\mathfrak{m}')^2}) \to \operatorname{gr}(A').$$

The next lemma gives $\operatorname{ht}(\ker(\gamma')) = \operatorname{ht}(\ker(\gamma))$ and the assertion follows. $\square$

Lemma 2.5.11. *Let $P = k[x_i \mid i \in I]$ and $P' = P \otimes_k k' = k'[x_i \mid i \in I]$, where $k \subset k'$ is a field extension. Then for every ideal $\mathfrak{a} \subset P$ we have $\operatorname{ht}(\mathfrak{a}) = \operatorname{ht}(\mathfrak{a}P')$.*

Proof. For short, let $\mathfrak{a}' = \mathfrak{a}P'$. If I is finite, then the lemma follows from dimension theory. In general, suppose by contradiction that $\operatorname{ht}(\mathfrak{a}) \neq \operatorname{ht}(\mathfrak{a}')$. Then we can find a finite subset $J \subset I$ such that $\operatorname{ht}(\mathfrak{a}_J) \neq \operatorname{ht}(\mathfrak{a}'_J)$ (cf. Remark 1.6.11). Since $\mathfrak{a}'_J = \mathfrak{a}_J P'_J$, this contradicts the finite dimensional case. $\square$

2.5.2 Formal embedding codimension

In the case of local quasi-preadic rings which are *equicharacteristic*, looking at the completion instead of the associated graded provides a different way of defining embedding codimension. To distinguish the two, we introduce the following terminology.

Definition 2.5.12. The *formal embedding codimension* of an equicharacteristic local quasi-preadic ring $(A, \mathfrak{m}, k)$ is defined to be

$$\operatorname{fcodim}(A) := \inf_{\tau} \operatorname{ht}(\ker(\tau))$$

where the infimum is taken over all choices of formal embeddings $\tau \colon \widehat{P} \to \widehat{A}$ (see Definition 1.3.32).

Proposition 2.5.13. *In the above definition, we have* $\operatorname{fcodim}(A) = \operatorname{ht}(\ker(\tau))$ *for every efficient formal embedding $\tau \colon \widehat{P} \to \widehat{A}$.*

Proof. Given two formal embeddings $\tau \colon \widehat{P} \to \widehat{A}$ and $\tau' \colon \widehat{P'} \to \widehat{A}$ with τ efficient, by Remark 1.3.37 there is a surjection $\varphi \colon \widehat{P'} \to \widehat{P}$ such that $\tau' = \tau \circ \varphi$, and hence $\operatorname{ht}(\ker(\tau')) \geq \operatorname{ht}(\ker(\tau))$. $\square$

Remark 2.5.14. If A is a local quasi-preadic k-algebra such that the residue field $A/\mathfrak{m}$ is separable over k, then it follows by Corollary 1.3.35 that the following are equivalent:

1. A is formally smooth over k.

2. $\operatorname{ecodim}(A) = 0$.

3. $\operatorname{fcodim}(A) = 0$.

Proposition 2.5.15. *Let $(A, \mathfrak{m}, k)$ be an equicharacteristic local quasi-preadic ring, then*

$$\operatorname{edim}(A) \geq \dim(\widehat{A}) + \operatorname{fcodim}(A),$$

and equality holds if A has finite embedding dimension. In particular, if A is Noetherian preadic then $\operatorname{edim}(A) = \dim(A) + \operatorname{fcodim}(A)$.

Proof. Consider an efficient formal embedding $\tau\colon \widehat{P} = k[[x_i \mid i \in I]] \to \widehat{A}$. The first formula follows from the simple fact that $\dim(\widehat{P}) \geq \operatorname{ht}(\ker(\tau)) + \dim(\widehat{P}/\ker(\tau))$. If A has finite embedding dimension then the set I is finite, and equality holds in the formula because a power series ring in finitely many variables is catenary of dimension equal to the number of variables. The second formula follows from the first and the fact that $\dim(A) = \dim(\widehat{A})$ if A is Noetherian preadic. □

Corollary 2.5.16. *If $(A, \mathfrak{m}, k)$ is an equicharacteristic local quasi-preadic ring of finite embedding dimension, then*

$$\operatorname{ecodim}(A) = \operatorname{fcodim}(A).$$

Proof. By Propositions 2.5.2 and 2.5.15, it suffices to show that $\dim(\operatorname{gr}(A)) = \dim(\widehat{A})$. By Corollary 1.3.35 the completion $\widehat{A}$ is the quotient of a power series ring in finitely many variables, and therefore is Noetherian and carries the $\widehat{\mathfrak{m}}$-adic topology. The result now follows from [Mat89, Theorem 15.7] and the identification $\operatorname{gr}(\widehat{A}) \simeq \operatorname{gr}(A)$. □

Recall from Definition 1.6.2 that a *DGK decomposition* of an equicharacteristic local quasi-preadic ring $(A, \mathfrak{m}, k)$ is an isomorphism $\widehat{A} \simeq \widehat{B} \mathbin{\widehat{\otimes}}_k \widehat{P}$, where $(B, \mathfrak{n}, k)$ is a local preadic Noetherian ring and $\widehat{P} = k[[x_i \mid i \in I]]$.

Proposition 2.5.17. *Let $(A, \mathfrak{m}, k)$ be an equicharacteristic local quasi-preadic ring. If A admits a DGK decomposition $\widehat{A} \simeq \widehat{B} \mathbin{\widehat{\otimes}}_k \widehat{P}$, then*

$$\operatorname{ecodim}(A) = \operatorname{fcodim}(A) = \operatorname{ecodim}(B) < \infty.$$

Proof. Since B is Noetherian, we have $\operatorname{ecodim}(B) < \infty$. By Corollary 2.5.16, we have that $\operatorname{ecodim}(B) = \operatorname{fcodim}(B)$. Moreover, Remark 1.6.3 and Theorem 1.6.9 together imply that $\operatorname{fcodim}(B) = \operatorname{fcodim}(A)$. It remains to shows that $\operatorname{ecodim}(A) = \operatorname{ecodim}(B)$.

To that avail, let $\mathfrak{n}$ be the maximal ideal of B and $\gamma_B\colon \operatorname{Sym}_k(\mathfrak{n}/\mathfrak{n}^2) \to \operatorname{gr}(B)$ the natural map. By Proposition 1.2.21 we have a diagram

$$\begin{tikzcd}
\operatorname{Sym}_k(\mathfrak{m}/\overline{\mathfrak{m}^2}) \arrow[r, "\gamma"] \arrow[d, "\simeq"'] & \operatorname{gr}(A) \arrow[d, "\simeq"] \\
\operatorname{Sym}_k(\mathfrak{n}/\mathfrak{n}^2) \otimes_k P \arrow[r, "\gamma_B \otimes \mathrm{id}_P"'] & \operatorname{gr}(B) \otimes_k P.
\end{tikzcd}$$

Let $\mathfrak{n}$ be the maximal ideal of B and $\gamma_B\colon \operatorname{Sym}_k(\mathfrak{n}/\mathfrak{n}^2) \to \operatorname{gr}(B)$ the canonical map. Write $P_A = \operatorname{Sym}_k(\mathfrak{m}/\mathfrak{m}^2)$, $P_B = \operatorname{Sym}_k(\mathfrak{n}/\mathfrak{n}^2)$, $\mathfrak{a} = \ker(\gamma) \subset P_A$, and $\mathfrak{b} = \ker(\gamma) \subset P_B$. The DGK decomposition gives isomorphisms $\operatorname{gr}(A) \simeq \operatorname{gr}(B) \otimes_k P$ and $P_A \simeq P_B \otimes_k P$. Moreover $\gamma = \gamma_B \otimes \mathrm{id}_P$, and therefore $\mathfrak{a} = \mathfrak{b}P_A$. Since P_B is a polynomial ring in finitely many variables, we see that $\operatorname{ht}(\mathfrak{a}) = \operatorname{ht}(\mathfrak{b})$, and the result follows. □

Remark 2.5.18. The analogous statement of Proposition 2.5.17 holds for equicharacteristic local quasi-preadic rings $(A, \mathfrak{m}, k)$ admitting a weak DGK decomposition.

The proof of Corollary 2.5.16 does not extend beyond the case of finite embedding dimension. Nonetheless, the following general comparison theorem holds.

Theorem 2.5.19. *For every equicharacteristic quasi-preadic local ring $(A, \mathfrak{m}, k)$, we have*

$$\operatorname{ecodim}(A) \leq \operatorname{fcodim}(A).$$

Proof. Fix an efficient formal embedding $\tau\colon \widehat{P} \to \widehat{A}$, and let $\operatorname{gr}(\tau)\colon P \to \operatorname{gr}(A)$ be the induced map on associated graded rings (here we identify $\operatorname{gr}(\widehat{P}) = \operatorname{gr}(P) = P$). We have $P \simeq \operatorname{Sym}_k(\mathfrak{m}/\mathfrak{m}^2)$ and $\operatorname{gr}(\tau)$ gets identified with the canonical surjection γ. In particular, it is enough to show that

$$\operatorname{ht}(\ker(\tau)) \geq \operatorname{ht}(\ker(\operatorname{gr}(\tau))).$$

Write $\mathfrak{a} = \ker(\tau)$. By [Bou72, Chapter III, Section 2.4, Proposition 2], we have that $\ker(\operatorname{gr}(\tau)) = \operatorname{in}(\mathfrak{a})$. To conclude, it is therefore enough to prove that

$$\operatorname{ht}(\mathfrak{a}) \geq \operatorname{ht}(\operatorname{in}(\mathfrak{a})).$$

This follows from the next proposition. □

Proposition 2.5.20. *Let $P = k[x_i \mid i \in I]$ and $\widehat{P} = k[[x_i \mid i \in I]]$, where k is a field. Let $\mathfrak{a} \subset \widehat{P}$ be an ideal and $\operatorname{in}(\mathfrak{a}) \subset P$ the corresponding initial ideal. Then* $\operatorname{ht}(\mathfrak{a}) \geq \operatorname{ht}(\operatorname{in}(\mathfrak{a}))$.

Proof. Let $J \subset I$ be a finite subset such that $\operatorname{in}(\mathfrak{a})_J$ has the same height of $\operatorname{in}(\mathfrak{a})$ if $\operatorname{ht}(\operatorname{in}(\mathfrak{a})) < \infty$, or has arbitrary large height if $\operatorname{ht}(\operatorname{in}(\mathfrak{a})) = \infty$. Let $c := \operatorname{ht}(\operatorname{in}(\mathfrak{a})_J)$.

By [BH93, Proposition 1.5.11], we can fix homogeneous elements $g_1, \dots, g_c \in \operatorname{in}(\mathfrak{a})_J$ forming a regular sequence in P_J. By the definition of initial ideal, there are elements $f_1, \dots, f_c \in \mathfrak{a}$ such that $\operatorname{in}(f_i) = g_i$ for all i. Let $R = R(\widehat{P}) := \bigoplus_{n \in \mathbb{Z}} \widehat{\mathfrak{m}^n} u^{-n}$ be the extended Rees algebra of $\widehat{P}$, where we set $\widehat{\mathfrak{m}^n} = \widehat{P}$ whenever $n < 0$. For every i, let $\widetilde{f_i} := u^{-\operatorname{ord}(f_i)} f_i \in R$. Note that $\widetilde{f_i}|_{u=0} = \operatorname{in}(f_i) = g_i$ via the identification $R/uR \simeq P$.

We claim that, for every $1 \leq r \leq c$, the elements $\widetilde{f}_1, \dots, \widetilde{f}_r$ form a regular sequence in R and $R/(\widetilde{f}_1, \dots, \widetilde{f}_r)$ is flat over $k[u]$. We argue by induction on r, the assertion being clear if $r = 0$. Letting for short $B := R/(\widetilde{f}_1, \dots, \widetilde{f}_{r-1})$, we know by induction that B is flat over $k[u]$. Assume that there exists $h \in \widehat{P}$ with $h = \sum_{i=1}^{r-1} a_i f_i$ and $\operatorname{in}(h)$ not divisible by $g_1, \dots, g_{r-1}$. Writing

$$\sum_{i=1}^{r-1} u^{\operatorname{ord}(f_i)} a_i \widetilde{f}_i = u^{\operatorname{ord}(h)} \widetilde{h},$$

Lemma 2.5.21 yields that B has torsion over $k[u]$, which gives a contradiction. Thus $\mathrm{in}(f_1, \dots, f_{r-1}) = (g_1, \dots, g_{r-1})$ and B is isomorphic to the algebra $\bigoplus_{n \in \mathbb{Z}} \mathfrak{b}_n u^{-n}$ where $\mathfrak{b}_n := (\widehat{\mathfrak{m}^n} + (f_1, \dots, f_{r-1}))/(f_1, \dots, f_{r-1})$ for $n \geq 0$ and $\mathfrak{b}_n = B$ for $n < 0$. It follows by Proposition 1.2.16 that $\bigcap_{n \geq 1} \mathfrak{b}_n = \{0\}$. Then Lemma 2.5.22 implies that B is (u)-adically separated, and Lemma 2.5.23 (with $t = u$) implies that the class b of $\widetilde{f_r}$ in B is a regular element and B/bB is flat over $k[u]$.

The natural isomorphism $R/(u-1)R \simeq \widehat{P}$ sends $\widetilde{f_i}$ to f_i, and hence we see by Lemma 2.5.23 (with $t = u - 1$) that $f_1, \dots, f_c$ form a regular sequence in $\widehat{P}$. This implies that $\mathrm{depth}(\mathfrak{a}, \widehat{P}) \geq c$. We conclude using the fact that $\mathrm{ht}(\mathfrak{a}) \geq \mathrm{depth}(\mathfrak{a}, \widehat{P})$ (e.g., see [AT09, Proposition 2.3 and Lemma 3.2]). □

Lemma 2.5.21. *Let $f_1, \dots, f_r \in \widehat{P}$. If $h \in \widehat{P}$, then $\mathrm{in}(h) \in (\mathrm{in}(f_1), \dots, \mathrm{in}(f_r))$ if and only if there exist elements $b_1, \dots, b_r$ in the Rees algebra $R = R(\widehat{P})$ such that*

$$\widetilde{h} = \sum_{i}^{r} b_i \widetilde{f_i}.$$

Proof. Given $\widetilde{h} = \sum b_i \widetilde{f_i}$, we may assume that b_i is homogeneous in R, i.e. of the form $b_i = u^{-\mathrm{ord}(h) + \mathrm{ord}(f_i)} a_i$ with $a_i \in \widehat{P}$. But then $\mathrm{ord}(a_i) \geq \mathrm{ord}(h) - \mathrm{ord}(f_i)$ and the claim follows. □

Lemma 2.5.22. *Let A be a ring and $(\mathfrak{a}_n)_{n \geq 0}$ a graded sequence of ideals of A, and let $R(A) := \bigoplus_{n \in \mathbb{Z}} \mathfrak{a}_n u^{-n}$ where we set $\mathfrak{a}_n = A$ for $n < 0$. Assume that $\bigcap_{n \geq 1} \mathfrak{a}_n = \{0\}$. Then $R(A)$ is (u)-adically separated.*

Proof. Let $a \in R(A)$ be any element. Write $a = \sum_{i=p}^{q} a_i u^{-i}$ for some $a_i \in \widehat{P}$ and $p, q \in \mathbb{Z}$. By the definition of Rees algebra, we have $a_i \in \mathfrak{a}_i$ for all i. The condition that $a \in u^n R(A)$ is equivalent to having $a_i \in \mathfrak{a}_{n+i}$ for all i. If $a \in \bigcap_{n \geq 1} u^n R(A)$, then we have $a_i \in \bigcap_{n \geq 1} \mathfrak{a}_n = \{0\}$ for all i, and hence $a = 0$. □

Lemma 2.5.23. *Let B be a flat and $k[t]$-algebra. For any given $b \in B$, consider the following properties:*

1. *b is a regular element of B and B/bB is flat over $k[t]$.*
2. *The image $\bar{b}$ of b in B/tB is regular.*

Then (1) ⇒ (2), and the converse holds if B is (t)-adically separated.

Proof. The proof is an adaptation of the proof of [Mat89, Theorem 22.5]. The implication (1) ⇒ (2) follows by the snake lemma applied to the commutative diagram

$$\begin{tikzcd}
B \arrow[r, "b"] \arrow[d, "t"] & B \arrow[r] \arrow[d, "t"] & B/bB \arrow[d, "t"] \\
B \arrow[r, "b"] \arrow[d] & B \arrow[r] \arrow[d] & B/bB \\
B/tB \arrow[r, "\bar{b}"] & B/tB &
\end{tikzcd}$$

after observing that the map $B \to B$ given by multiplication by b is injective since b is regular, and so is the map $B/bB \to B/bB$ given by multiplication by t since B/bB is flat over $k[t]$.

In order to prove the implication (2) $\Rightarrow$ (1) when B is (t)-adically separated, suppose $x \in B$ is an element such that $bx = 0$. Then $\bar{b}\bar{x} = 0$ in B/tB and hence $\bar{x} = 0$. This means that $x \in tB$. Suppose $x \in t^n B$ for some positive integer n, and write $x = t^n y$ in B. Then $t^n(by) = bx = 0$ and hence $by = 0$ since B is flat over $k[t]$. This implies that $y \in tB$, and hence $x \in t^{n+1}B$. Therefore $x \in \bigcap_{n\geq 1} t^n B$, and since B is (t)-adically separated, this means that $x = 0$. This proves that b is a regular element. To conclude that B/bB is flat over $k[t]$, we just compute that $\mathrm{Tor}_1^{k[t]}(k, B/bB) = 0$ from the exact sequence $0 \to B \to B \to B/bB \to 0$ and apply [Mat89, Theorem 22.3]. $\square$

Question 2.5.24. We do not know of any example where the inequality in Theorem 2.5.19 is strict. The question whether $\mathrm{ecodim}(A) = \mathrm{fcodim}(A)$ holds for all equicharacteristic local (quasi-preadic) rings $(A, \mathfrak{m}, k)$ is, to our knowledge, still open.

Chapter 3

Arc spaces and their singularities

In this chapter we introduce the space of arcs X_∞ of an algebraic variety X, and use the methods established in Chapters 1 and 2 to study the singularities of X_∞. One of the main results in this direction is Theorem 3.4.2, which characterizes non-degenerate k-arcs as those with finite embedding dimension, with an explicit bound provided. In Section 3.5 we give a detailed comparison of the geometric approach behind our results to the Drinfeld–Grinberg–Kazhdan theorem (see Theorem 3.4.1). Finally, in Section 3.6 we prove some consequences involving the Mather–Jacobian discrepancy.

Section 3.2 incorporates parts of both [CH17] and [CN20], whereas Sections 3.3 to 3.6 were taken from [CdFD20].

3.1 Jets and arcs

In this section we aim to introduce jet and arc spaces as well as state some well-known results describing their geometry, which we will make use of in later sections. We shall note that this is not intended as a comprehensive introduction to the theory of arc spaces; for that, we refer the reader to the plethora of excellent surveys: [Mus07] for an introduction to arc spaces in the context of birational geometry; [Ish07; Ish12; dFe18] from the point of view of the Nash problem and finally [DL01; Vey06] for an introduction in the flavor of motivic integration.

Throughout this section we will use the same abbreviated notation as in Sections 2.1 and 2.2 and write $R[[t]]_n := R[t]/(t^{n+1})$ for $n \in \mathbb{N}$ and $R[[t]]_\infty := R[[t]]$.

3.1.1 Jet and arc spaces

Let X be a scheme over a base ring k. For $n \in \mathbb{N}$, an *n-jet* on X is a morphism

$$\alpha : \operatorname{Spec}(L[[t]]_n) \to X,$$

with $L \supset k$ a field extension. An *arc* on X is a morphism

$$\alpha : \operatorname{Spec}(L[[t]]) \to X.$$

We write 0 for the unique closed point of $\mathrm{Spec}(L[[t]]_n)$ and η for the generic point of $\mathrm{Spec}(L[[t]])$, as well as $\alpha_0 := \alpha(0)$ and $\alpha_\eta := \alpha(\eta)$. Note that a 1-jet α on X is nothing but a tangent vector to X at α_0.

If we consider $L[[t]]_n$ as a topological ring with respect to the t-adic topology, then any n-jet resp. arc on X corresponds uniquely to a map

$$\mathrm{Spf}(L[[t]]_n) \to \widehat{X}_{\alpha_0},$$

where $\widehat{X}_{\alpha_0}$ denotes the formal neighborhood of X at α_0.

We also define the n-jet functor of X as

$$Y \in \mathrm{Sch}_k \mapsto \mathrm{Hom}_k(Y \times_k \mathrm{Spec}(k[[t]]_n), X).$$

The *n-th jet space* X_n of X is the k-scheme representing the jet functor, that is,

$$\mathrm{Hom}_k(Y, X_n) \simeq \mathrm{Hom}_k(Y \times_k \mathrm{Spec}(k[[t]]_n), X).$$

In particular, if $L \supset k$ is a field extension, then we have that the L-points of X_n are just the n-jets on X with coefficients in L; written

$$X_n(L) \simeq X(L[[t]]_n).$$

For $m \geq n$, the truncations $k[[t]]_m \to k[[t]]_n$ induce natural morphisms $\pi_{m,n} : X_m \to X_n$ which form a projective system. The *arc space* X_∞ of X is then defined to be $X_\infty = \lim_n X_n$. By definition, the arc space X_∞ represents the functor

$$Y \in \mathrm{Sch}_k \mapsto \mathrm{Hom}_k(Y \widehat{\times}_k \mathrm{Spf}(k[[t]]), X),$$

where $Y \widehat{\times}_k \mathrm{Spf}(k[[t]])$ is the formal completion of the product of schemes $Y \times_k \mathrm{Spec}(k[[t]])$. In particular, for a field extension $L \supset k$ we get

$$X_\infty(L) \simeq X(L[[t]]).$$

If $X = \mathrm{Spec}(A)$ is affine, then it follows that for $n \in \mathbb{N}$ the jet scheme X_n represents the functor

$$R \in \mathrm{Alg}_k \mapsto \mathrm{Hom}_k(\mathrm{Spec}(R[[t]]_n), X) \simeq \mathrm{Hom}_k(A, R[[t]]_n),$$

and thus, by the results of Section 2.1, we have $X_n = \mathrm{Spec}(A_n)$, where $A_n = \mathbb{HS}^n_k(A)$ is the n-th Hasse-Schmidt algebra of A. Moreover, since $\mathbb{HS}^\infty_k(A) = \mathrm{colim}_n \mathbb{HS}^n_k(A)$ we have that $X_\infty = \mathrm{Spec}(\mathbb{HS}^\infty_k(A))$.

Example 3.1.1. Let $X = \mathbb{A}^d_k = \mathrm{Spec}(k[x_1, \dots, x_d])$. Then clearly, for $n \in \mathbb{N}$,

$$X_n = \mathrm{Spec}(k[x_{1,j}, \dots, x_{d,j} \mid j = 0, \dots, n]) = \mathbb{A}^{d(n+1)}_k.$$

In particular, $X_n \simeq X \times \mathbb{A}^{dn}_k$. Moreover, we see that for $d > 0$ the scheme X_∞ is non-Noetherian and of infinite dimension.

Example 3.1.2. Let $X = \mathrm{Spec}(k[x, y]/(y^2 - x^3))$ be the plane cusp singularity. Then X_n is defined as a subscheme of $(\mathbb{A}^2_k)_n$ by the system of equations

$$\begin{gathered} y_0^2 - x_0^3 = 0 \\ 2y_0y_1 - 3x_0^2x_1 = 0 \\ y_1^2 + 2y_0y_2 - 3x_0x_1^2 - 3x_0^2x_2 = 0 \\ \cdots \end{gathered}$$

To give a construction of X_n for general k-schemes X we may proceed as in [Voj07] and construct on X a sheaf of $\mathcal{O}_X$-algebras $\mathcal{H}_n$ such that, for each affine open $U = \mathrm{Spec}(A) \subset X$, we have that

$$\Gamma(U, \mathcal{H}_n) \simeq \mathbb{HS}^n_k(A).$$

Then we obtain X_n by taking the relative spectrum of the $\mathcal{O}_X$-algebra $\mathcal{H}_n$.

Remark 3.1.3. If k is a field of characteristic 0, then we may describe X_∞ in terms of differential algebra (see [Kol73]). Assuming X is an affine subscheme of $\mathbb{A}^d_k = \mathrm{Spec}(k[x_1, \dots, x_d])$, we define a derivation D on the infinite-variate polynomial ring

$$k[x_{1,j}, \dots, x_{d,j} \mid j \in \mathbb{N}]$$

by defining $D(x_{i,j}) = x_{i,j+1}$. If X is defined by an ideal $\mathfrak{a}$ inside $k[x_1, \dots, x_d]$, then X_∞ is given by the *differential closure* of $\mathfrak{a}$ with respect to D. In fact, using Taylor expansion we obtain the description of Example 3.1.1 by applying the automorphism given by

$$x_{i,j} \mapsto \frac{1}{j!} x_{i,j}.$$

Remark 3.1.4. As a word of warning: while, for a k-scheme X, it is essentially a tautology that the jet space X_n represents the functor

$$R \mapsto \mathrm{Hom}_k(\mathrm{Spec}(R[[t]]/(t^{n+1})), X),$$

the same is a priori not clear for the space of arcs X_∞ in the case where X is not affine. In fact, this was proven in [Bha16] using methods of derived algebraic geometry.

Let us also describe in detail how to obtain a description of the formal neighborhood of an n-jet resp. an arc in terms of *deformations*. Recall from Section 1.1.4 that a *test ring* is a local k-algebra $(R, \mathfrak{m})$ such that $R/\mathfrak{m} = k$ and $\mathfrak{m}^n = 0$ for some n. The category of test rings was denoted by Nil_k.

Definition 3.1.5. Let X be a scheme over a field k and $n \in \mathbb{N} \cup \{\infty\}$. Let $\alpha \in X_n(k)$. If $(R, \mathfrak{m})$ is a test ring, then we write $u : R[[t]] \to k[[t]]$ for the t-linear extension of the quotient map $R \to R/\mathfrak{m} \simeq k$. A *$R$-deformation* of α is a diagram

$$\begin{tikzcd} \mathrm{Spec}(R[[t]]_n) \arrow[r, "\tilde{\alpha}"] & X . \\ \mathrm{Spec}(k[[t]]_n) \arrow[u, "u^*"] \arrow[ur, "\alpha"'] & \end{tikzcd}$$

Remark 3.1.6. Since R is discrete with respect to its $\mathfrak{m}$-adic topology, an R-deformation of α is equivalent to a diagram

$$\begin{tikzcd} \mathrm{Spf}(R[[t]]_n) \arrow[r, "\tilde{\alpha}"] & \widehat{X}_{\alpha_0} , \\ \mathrm{Spf}(k[[t]]_n) \arrow[u] \arrow[ur, "\alpha"'] & \end{tikzcd}$$

where $\widehat{X}_{\alpha_0}$ denotes the formal neighborhood of X at α_0.

By Corollary 1.1.36, the formal neighborhood $\widehat{X}_x$ of a k-scheme X at a k-rational point $x \in X(k)$ is determined by its R-deformations. Together with the definition of X_n we obtain the following characterization (used for example in [Dri02] to prove Theorem 3.4.1):

Lemma 3.1.7. *Let X be a scheme over a field k and $n \in \mathbb{N} \cup \{\infty\}$. Let $\alpha \in X_n(k)$. If $\widehat{(X_n)}_\alpha$ denotes the formal neighborhood of α, then, for every test ring R, we have*

$$\begin{aligned}\widehat{(X_n)}_\alpha(R) &\simeq \{R\text{-deformations } \tilde{\alpha} : \operatorname{Spec}(R[[t]]_n) \to X \text{ of } \alpha\} \\ &\simeq \{R\text{-deformations } \tilde{\alpha} : \operatorname{Spf}(R[[t]]_n) \to \widehat{X}_{\alpha_0} \text{ of } \alpha\}.\end{aligned}$$

3.1.2 Geometry of jet and arc spaces

In this section our aim is to state some fundamental geometric properties of jet and arc spaces. We start with the following lemma.

Lemma 3.1.8. *Let $f : X \to Y$ be formally étale. Then, for all $n \in \mathbb{N} \cup \{\infty\}$, we have that the diagram*

$$\begin{tikzcd} X_n \arrow[r, "f_n"] \arrow[d] & Y_n \arrow[d] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

is Cartesian.

Recall that, if X is a smooth scheme over a field k, then for each $p \in X$ there exists an open subscheme $U \subset X$ and an étale morphism $U \to \mathbb{A}^d$, where $d = \dim_p X$. Thus, from Example 3.1.1 together with Lemma 3.1.8 we obtain the following result on the structure of the natural maps $\pi_{n+1,n} : X_{n+1} \to X_n$.

Corollary 3.1.9. *Let X be a smooth scheme over a field k of dimension d. Then $\pi_{n+1,n} : X_{n+1} \to X_n$ is a locally trivial fibration with fiber $\mathbb{A}^d$; that is, there exists a covering of X_n by opens U such that $\pi_{n+1,n}$ restricted to each U is just* $\mathrm{pr}_1 : U \times \mathbb{A}^d \to U$.

In particular we have:

Corollary 3.1.10. *Let X be a smooth scheme over a field k. Then*

1. *For each $n \in \mathbb{N}$ the jet space X_n is smooth over k.*
2. *The arc space X_∞ is formally smooth over k.*

Remark 3.1.11. In [Ish09] a strong converse to the above was proven: for X to be smooth it is sufficient that there exists an $n \in \mathbb{N}$ such that X_n is smooth over k. Additionally, it was shown that, if k is algebraically closed and of characteristic 0, then X is smooth over k if and only if there exists $n > m > 0$ such that $\pi_{n,m} : X_n \to X_m$ is flat.

Remark 3.1.12. In [Ish11] the same phenomenon was shown to be true for a host of other properties. Namely, if X_n has property (P) for some $n \in \mathbb{N}$, then so does X, where (P) can be reduced, irreducible, connected, normal, locally complete intersection, $\mathbb{Q}$-factorial, $\mathbb{Q}$-Gorenstein, (log) canonical or (log) terminal. The converse was shown to be false for many of these properties. This indicates that, heuristically speaking, requiring X_n to have a certain property tends to be a rather strong assumption. See also Theorem 3.1.18 and Example 3.1.19.

Remark 3.1.13. In contrast to Remark 3.1.11, in [BS17a] the following was shown. Let X be a reduced, positive-dimensional scheme of finite type over a field k and let $\alpha : \mathrm{Spec}(k[[t]]) \to X$ be an arc such that $\alpha_\eta \in X_{\mathrm{sm}}$. Then $\widehat{X}_\alpha \simeq \mathrm{Spf}(k[[t_i \mid i \in \mathbb{N}]])$ if and only if the unique formal branch of X at α_0 containing $\mathrm{Im}(\alpha)$ is smooth. By Corollary 1.3.35, we see that this gives a characterization of formally smooth k-points of X_∞. Note here that we will show in Section 3.5 that a degenerate arc cannot be a formally smooth point of X_∞.

We now turn our attention to describing the basic topological properties of jet and arc spaces. The following theorem was first proven by Kolchin in [Kol73] in the setting of differential algebra. A later proof with a more geometric flavor was given in [NS05].

Theorem 3.1.14 (Kolchin's irreducibility theorem)**.** *Let X be a scheme over a field k of characteristic* 0*. If X is irreducible, then so is X_∞.*

Remark 3.1.15. Note that even if X is reduced, X_∞ is not reduced in general. In fact, in [Seb11] it was proven that, for a plane curve C over a field of characteristic 0, its arc space X_∞ is reduced if and only if C is smooth over k. The proof of this statement used derivations to explicitly give a nontrivial nilpotent element of $\mathcal{O}_{C_\infty}$.

While Theorem 3.1.14 is no longer true in positive characteristics, there still exists a reasonable description of the irreducible components of X_∞ over arbitrary fields, provided that X itself is of finite type. The following result was first proven over perfect fields in [Reg09] and in the general case in [NS10].

Theorem 3.1.16 ([NS10, Theorem 3.15])**.** *Let X be a scheme of finite type over a field k and write $Z := X \setminus X_{\mathrm{sm}}$, where X_{sm} denotes the locus of smooth points of X. Then:*

1. *If X is irreducible, then so is $X_\infty \setminus Z_\infty$.*
2. *There exists a bijection between those irreducible components of X which are geometrically reduced, and the irreducible components of $X_\infty \setminus Z_\infty$.*

In particular, if k is perfect and X reduced, then the components of X are in bijection with those of $X_\infty \setminus Z_\infty$, recovering [Reg09, Theorem 2.9]. By induction on the dimension of X we get that X_∞ has only finitely many components. Arguing as in [NS10] we can generalize to arbitrary fields as:

Corollary 3.1.17 ([NS10, Corollary 3.16])**.** *If X is a scheme of finite type over a field k, then X_∞ has only finitely many irreducible components.*

One should remark that the situation is markedly different for jet spaces, as even for irreducible X over a field of characteristic 0 the jet spaces X_n may have several components. In fact, in [Mus01] the following remarkable result was proven:

Theorem 3.1.18 ([Mus01, Theorem 0.1]). *Let X is a variety over an algebraically closed field of characteristic* 0 *and assume that X is a local complete intersection. Then X_n is irreducible for all n if and only if X has rational singularities.*

Example 3.1.19. Let k be a field of characteristic 0 and $C \subset \mathbb{A}_k^2$ be a plane curve with $\mathrm{Sing}(C) = \{0\}$; for example $C = \mathrm{Spec}(k[x,y]/(y^2 - x^3))$. Then the first jet scheme C_1 is just the Zariski tangent bundle of C and thus has two components, with one given by $\pi^{-1}(\{0\})$, where $\pi : C_1 \to C$ is the canonical projection.

3.2 Differentials on jet and arc spaces

One of the more comprehensive studies of the schematic structure of jet and arc spaces - and in particular of their singularities - has been conducted in [dFD20], whose main result is a formula for the sheaf of differentials, which in the affine case was proven in this thesis in Section 2.2. From this formula several results concerning the embedding dimension of jets and arcs were obtained, which in turn gave new proofs of several key results such as a variant of the Birational Transformation Rule, as proven in [DL99a]. The purpose of this section is to globalize the results of Section 2.2 to give the sheaf-theoretic version of Theorem 2.2.15 and to summarize several consequences derived in [dFD20] which will be used in the latter sections of this chapter. Finally, we will prove a result describing the formal neighborhood of degenerate arcs whose proof makes use of Theorem 2.2.15.

3.2.1 Sheaves of differentials on jet and arc spaces

This section should be read as a continuation of Section 2.2. There we established the existence of the Hasse-Schmidt module, which parametrizes higher derivations of modules. In particular, one consequence is the following formula, as first proven in [dFD20]:

Theorem (Theorem 2.2.15). *Let A be a k-algebra and, for $n \in \mathbb{N} \cup \{\infty\}$, let $A_n := \mathbb{HS}_k^n(A)$ be the n-th Hasse-Schmidt algebra. Then there exists an (A, A_n)-bimodule Q_n, free over A_n, such that*

$$\Omega_{A_n/k} \simeq \Omega_{A/k} \otimes_A Q_n$$

as A_n-modules.

In the same way as jet and arc spaces are obtained by gluing Hasse-Schmidt algebras, we will glue Hasse-Schmidt modules to obtain coherent sheaves on jet and arc spaces. In particular, this will allow us to give a global version of the above formula for the sheaf of differentials.

Throughout this section, we will assume that X is a scheme over $S = \mathrm{Spec}(k)$, where k is an arbitrary commutative ring. We note that the assumption that S is affine is not needed and only made for convenience's sake. As in Section 2.1 and Section 2.2, we will write $k[[t]]_n := k[t]/(t^{n+1})$.

Let $\mathcal{F}$ be a quasi-coherent sheaf on X, that is, there exists a covering by open affines $U_i = \mathrm{Spec}(A_i) \subset X$ such that $\mathcal{F}|_{U_i}$ corresponds to some A_i-module M_i. As before, set $(A_i)_n := \mathbb{HS}_k^n(A_i)$. Let $n \in \mathbb{N} \cup \{\infty\}$. Recall that the n-th

Hasse-Schmidt module $\mathbb{HS}^n_{A_i/k}(M_i)$ of M_i was defined to be the $(A_i)_n$-module representing the functor

$$N \mapsto \mathrm{Hom}_{A_i}(M_i, N[[t]]_n).$$

The n-th jet space resp. arc space X_n of X has a covering by open affines $(U_i)_n = \mathrm{Spec}((A_i)_n)$. For i, j the sheaf $\mathcal{F}$ gives rise to isomorphisms $(M_i)_f \simeq (M_j)_g$, where f and g run over partitions of unity of $U_i \cap U_j$. By Lemma 2.2.11 we have that

$$\mathbb{HS}^n_{A_i/k}(M_i) \otimes_{(A_i)_n} ((A_i)_n)_f \simeq \mathbb{HS}^n_{(A_i)_f/k}(M_i \otimes_{A_i} (A_i)_f),$$

and by functoriality of the Hasse-Schmidt module we have that $\mathbb{HS}^n_{A_i/k}(M_i)$ glue to an $\mathcal{O}_{X_n}$-module $\mathcal{F}_n$. To summarize:

Proposition 3.2.1. *Let X be a scheme over k and $\mathcal{F}$ a quasi-coherent sheaf of $\mathcal{O}_X$-modules. Let $n \in \mathbb{N} \cup \{\infty\}$. Then there exists a quasi-coherent sheaf $\mathcal{F}_n$ of $\mathcal{O}_{X_n}$-modules such that, if $U = \mathrm{Spec}(A) \subset X$ is an affine open, then*

$$\Gamma(U_n, \mathcal{F}_n) \simeq \mathbb{HS}^n_{A/k}(M),$$

where $M = \Gamma(U, \mathcal{F})$.

Consider now the case $n \in \mathbb{N}$. Setting $Q_n := \mathbb{HS}^n_{A/k}(A)$, from Theorem 2.2.5 we have that

$$\mathbb{HS}^n_{A/k}(M) \simeq M \otimes_A Q_n.$$

Recall that the A-action on Q_n is induced by the map $\gamma_A : A \to A_n[[t]]_n$. To globalize the above statement, we make use of the universal n-jet as introduced in Section 3.1. Namely, for any scheme X its n-the jet space comes equipped with a morphism $\gamma : X_n \times_k \mathrm{Spec}(k[[t]]_n) \to X$. Denote by ρ the canonical projection $X_n \times_k \mathrm{Spec}(k[[t]]_n) \to X_n$. If $\mathcal{G}$ is a quasi-coherent $\mathcal{O}_{X_n}$-module and $U = \mathrm{Spec}(A)$ an affine open of X such that $\mathcal{G} \mid_{U_n}$ corresponds to an A_n-module N, then

$$\Gamma(U, \gamma_* \rho^* \mathcal{G}) \simeq N[[t]]_n$$

as A-modules. On the other hand, we obtain a quasi-coherent sheaf $\mathcal{Q}_n$ on $X_n \times \mathrm{Spec}(k[[t]]_n)$ by gluing $Q_n = \mathbb{HS}^n_{A/k}(A)$ for $U = \mathrm{Spec}(A) \subset X$ open as above. Then it the following result is straightforward to verify:

Theorem 3.2.2. *Let X be a scheme over k and $\mathcal{F}$ a quasi-coherent $\mathcal{O}_X$-module. Let $n \in \mathbb{N}$ and $\mathcal{F}_n$ the sheaf obtained from $\mathcal{F}$ as in Proposition 3.2.1. Then $\mathcal{F}_n \simeq \rho_*(\gamma^* \mathcal{F} \otimes \mathcal{Q}_n)$. Moreover, $\mathcal{F}_n$ represents the functor*

$$\mathcal{G} \in \mathrm{Mod}_{\mathcal{O}_{X_n}} \mapsto \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \gamma_* \rho^* \mathcal{G}).$$

Remark 3.2.3. As $\mathcal{Q}_n$ is trivial for $n \in \mathbb{N}$ (see Remark 2.2.7), we have that $\mathcal{F}_n \simeq \rho_* \gamma^* \mathcal{F}$.

From Remark 2.2.8 we see that, if $\mathcal{F}$ is locally free then so is $\mathcal{F}_n$. Together with Theorem 2.2.12 this gives the following result on universal vector bundles on jet spaces:

Proposition 3.2.4. *Let X be a scheme over k and $n \in \mathbb{N}$. Then the assignment $\mathcal{F} \mapsto \mathcal{F}_n$ gives a functor $\mathrm{Coh}(X) \to \mathrm{Coh}(X_n)$ which preserves the property of being locally free. Moreover, if $E \to X$ is the vector bundle corresponding to a locally free $\mathcal{F} \in \mathrm{Coh}(X)$, then its jet space $E_n \to X_n$ is the vector bundle corresponding to $\mathcal{F}_n \in \mathrm{Coh}(X_n)$.*

Proof. The statement follows from the above and the fact that, if M is a finite free right A-module and N an (A, B)-bimodule which is finite free over B, then the tensor product $M \otimes_A N$ is finite free as a right B-module. □

The construction of the sheaf $\mathcal{F}_n$ was used in [dFD20] to provide a global variant of Theorem 2.2.15, as well as in [dFD19] to show how the jet space behaves under Nash blow-up. The following example shows that, even in the very easy case of a line bundle $\mathcal{L}$ on $\mathbb{P}^1$, its associated bundle $\mathcal{L}_n$ on the n-th jet space has interesting structure:

Example 3.2.5. Let $X = \mathbb{P}^1$ and and $\mathcal{L}$ an invertible sheaf on $\mathbb{P}^1$, isomorphic to $\mathcal{O}(d)$ for some $d \in \mathbb{Z}$. Let U_0 and U_1 be the standard affine opens of $\mathbb{P}^1$. We write $U_i = \mathrm{Spec}(k[t_i])$, then $\mathcal{L} \mid_{U_i}$ is generated by e_i and the transition map $\mathcal{L} \mid_{U_0 \cap U_1} \to \mathcal{L} \mid_{U_1 \cap U_0}$ is given by $e_0 \mapsto t_1^d e_1$. For $n \in \mathbb{N}$ we want to describe explicitly the sheaf $\mathcal{L}_n$ on $(\mathbb{P}^1)_n$. To that avail, note that $(\mathbb{P}^1)_n$ has an affine covering given by $(U_i)_n = \mathrm{Spec}(k[t_i^{(0)}, \dots, t_i^{(n)}])$. The restriction $\mathcal{L}_n \mid_{(U_i)_n}$ is freely generated by the sections $e_i^{(j)}$ for $j = 0, \dots, n$. By functoriality of the Hasse-Schmidt module the transition map is given by

$$e_0^{(j)} \mapsto \Delta_j \left((t_1^{(0)})^d e_1^{(j)} \right),$$

where Δ is the universal higher derivation associated to $\mathcal{L}_n \mid_{(U_1)_n}$. In particular, for $d = 1$, it is easy to see that the space of global sections of $\mathcal{O}(1)_n$ is generated by $e_i^{(j)}$, where $i = 0, 1$ and $j = 0, \dots, n$. This gives a notion of "coordinates" for $(\mathbb{P}^1)_n$. The same obviously holds for any projective space $\mathbb{P}^m$, $m \geq 1$.

Let us now turn our attention to the case $n = \infty$ and the equivalent statement to Theorem 3.2.2. The situation is slightly more subtle as the universal arc is given as

$$\gamma : X_\infty \widehat{\times}_k \mathrm{Spf}(k[[t]]) \to X,$$

which is a morphism between formal schemes. Similarly, the projection ρ should be thought of as a map of formal schemes as well. In particular, if $\mathcal{G}$ is a quasi-coherent sheaf on X_∞, then the pushforward $\gamma_* \rho^* \mathcal{G}$ will no longer be quasi-coherent over X. However, one can check that, by adapting the gluing for modules Q_∞, one obtains a quasi-coherent sheaf $\mathcal{Q}_\infty$ on $X_\infty \widehat{\times}_k \mathrm{Spf}(k[[t]])$, whose pushforward $\rho_* \mathcal{Q}_\infty$ is quasi-coherent over X_∞. We will skip the details here, but mention that this essentially follows from the structure of $\mathcal{Q}_\infty$ as the colimit of quasi-coherent sheaves. Then the following result holds:

Theorem 3.2.6. *Let X be a scheme over k and $\mathcal{F}$ a quasi-coherent $\mathcal{O}_X$-module. Let $\mathcal{F}_\infty$ be the quasi-coherent sheaf over X_∞ as in Proposition 3.2.1. Then*

$$\mathcal{F}_\infty \simeq \rho_*(\gamma^* \mathcal{F} \otimes \mathcal{Q}_\infty).$$

Finally, let us mention the the global version of Theorem 2.2.15:

Theorem 3.2.7. *Let X be a scheme over k and let $n \in \mathbb{N} \cup \{\infty\}$. Then*

$$\Omega_{X_n/k} \simeq \rho_*(\gamma^* \Omega_{X/k} \otimes \mathcal{Q}_n),$$

where γ is the universal n-jet resp. arc as defined above, and ρ is the corresponding projection onto X_n. If $n \in \mathbb{N}$, then moreover $\Omega_{X_n/k} \simeq \rho_* \gamma^* \Omega_{X/k}$.

3.2.2 The fiber over a jet and embedding dimension of jet spaces

One particularly useful application of Theorem 3.2.7 is, in conjuction with the structure theorem for modules over principal ideal rings, to compute the pullback of the sheaf of differentials along truncated arcs. This allows one to obtain formulas for the embedding dimension of truncated arcs as points in the jet space as well as bounds for the cotangent map of the truncation maps $X_\infty \to X_n$. These formulas were proven in [dFD20] and the goal of this section is to summarize the results contained in the latter which will be used in the upcoming sections. For proofs of the statements we will refer the reader to the original paper [dFD20].

Throughout this section let k denote any commutative ring. As in Section 3.2.1, if A is a k-algebra and $n \in \mathbb{N} \cup \{\infty\}$, then we write $A_n := \mathbb{HS}^n_k(A)$. If X is a scheme over k and $\alpha_n \in X_n$, then we denote by L_n the residue field of α_n and write $\dim(\alpha_n) := \operatorname{trdeg}_k(L_n)$. We may assume that $X = \operatorname{Spec}(A)$ is affine, then α_n corresponds to a map $A \to L_n[[t]]_n$, which, by abuse of notation, we will refer to by α_n as well. Write I_n for the prime ideal of A_n defining α_n. Then the Zariski tangent space at α_n is dual to I_n/I_n^2, and thus we have

$$\operatorname{edim}(\mathcal{O}_{X_n,\alpha_n}) = \dim_{L_n}(I_n/I_n^2).$$

If $\alpha \in X_\infty$ and $\alpha_n = \pi_n(\alpha)$ denotes the n-th trunctation of α, then

$$I_\infty/I_\infty^2 = \operatorname*{colim}_n \operatorname{Im}(I_n/I_n^2).$$

If M is an A-module, then we are interested in the pullback of M along α_n, that is, the $L_n[[t]]_n$-module

$$M \otimes_A L_n[[t]]_n.$$

Since $L_n[t]]_n$ is a principal ideal ring, the structure theorem for modules over such rings tells us that there exists a unique decomposition

$$M \otimes_A L_n[[t]]_n \simeq F \oplus T,$$

with F free and T torsion. Write $d(\alpha_n)$ for the rank of F; in [dFD20] this number is referred to as the *Betti number* of M with respect to α_n. If $m > n$ and $\alpha_n = \pi_{m,n}(\alpha_m)$, then $d(\alpha_n) \geq d(\alpha_m)$. Moreover, for $\alpha \in X_\infty$ we have

$$d(\alpha) = \dim_{k(\alpha_\eta)}(M \otimes_A k(\alpha_\eta)),$$

where α_η denotes the image under α of the generic point of $\operatorname{Spec}(L_n[[t]])$.

Making use of the formula for the sheaf of differentials established in Section 2.2 and the previous section the following structure theorem is obtained:

Theorem 3.2.8 ([dFD20, Theorem 7.2, Corollary 7.3]). *Let X be a scheme of finite type over k and $n \in \mathbb{N}$. Let $\alpha_n \in X_n$ and denote its residue field by L_n. Then there exists an isomorphism of $L_n[[t]]_n$-modules*

$$\Omega_{X_n/k} \otimes_{\mathcal{O}_{X_n}} L_n \simeq F \oplus T,$$

where T is torsion and F is free of rank $d_n = d(\alpha_n)$, the Betti number of $\Omega_{X/k}$ with respect to α_n. If $\alpha_n = \pi_n(\alpha)$ is the truncation of $\alpha \in X_\infty$, then

$$\dim_{L_n}(\Omega_{X_n/k} \otimes_{\mathcal{O}_{X_n}} L_n) = (n+1)d_n + \operatorname{ord}_\alpha(\operatorname{Fitt}^{d_n}(\Omega_{X/k})).$$

Proof. We may assume $X = \operatorname{Spec}(A)$ is affine and then apply Theorem 2.2.15 to obtain

$$\Omega_{A_n/k} \otimes_{A_n} L_n \simeq \Omega_{A/k} \otimes_A Q_n \otimes_{A_n} L_n.$$

Since $Q_n \simeq A_n[[t]]_n$ the result follows. □

Remark 3.2.9. Theorem 3.2.8 does not extend to arcs, since $Q_\infty \not\simeq A_n[[t]]$. However, the Betti number $d = d(\alpha)$ of $\Omega_{X/k}$ with respect to $\alpha \in X_\infty$ has a clear geometric meaning, namely, as remarked before it is equal to $\dim_{k(\alpha_\eta)}(\Omega_{X/k} \otimes k(\alpha_\eta))$. Note that for $n >> 0$ the Betti numbers d and $d(\alpha_n)$ for $\alpha_n = \pi_n(\alpha)$ agree.

The first application of Theorem 3.2.8 concerns the embedding dimension at truncated arcs and the dimension of the image of the cotangent map.

Lemma 3.2.10 ([dFD20, Lemma 8.1]). *Let X be an affine scheme of finite type over a perfect field k and let $\alpha \in X_\infty$. For $n \in \mathbb{N}$ denote by d_n the n-th Betti number of $\Omega_{X/k}$ with respect to the truncation $\alpha_n = \pi_n(\alpha)$. Then:*

$$\operatorname{edim}(\mathcal{O}_{X_n,\alpha_n}) = (n+1)d_n - \dim(\alpha_n) + \operatorname{ord}_\alpha(\operatorname{Fitt}^{d_n}(\Omega_{X/k})).$$

Lemma 3.2.11 ([dFD20, Lemma 8.3]). *Let $X = \operatorname{Spec}(A)$ be an affine scheme of finite type over a perfect field k, let $\alpha \in X_\infty$ with residue field L and for $n \in \mathbb{N}$ write $\alpha_n = \pi_n(\alpha)$. Let I_n and I_∞ be the ideals defining α_n resp. α and consider the map*

$$\lambda_n : I_n/I_n^2 \otimes_{A_n} L \to I_\infty/I_\infty^2$$

induced by the truncation π_n. Then

$$\dim_L(\operatorname{Im}(\lambda_n)) \geq (n+1)d - \dim(\alpha_n),$$

where $d = d(\alpha)$ is the Betti number of $\Omega_{X/k}$ with respect to α.

We also want to mention the following result, which in characteristic 0 follows directly from Kolchin's Irreducibility theorem (see Theorem 3.1.14):

Lemma 3.2.12 ([dFD20, Lemma 8.6]). *Let X be a scheme of finite type over a perfect field k, let $\alpha \in X_\infty$ and $Z \subset X$ the closed irreducible subset with generic point α_η. Then, for every $n \in \mathbb{N}$, we have:*

$$\dim(\alpha_n) \leq (n+1)\dim(Z).$$

We want to finish this section by mentioning the following characterization of *constructible* points of X_∞ not contained in $(\operatorname{Sing} X)_\infty$, that is, those $\alpha \in X_\infty$ whose closure $\overline{\{\alpha\}}$ is constructible. If X is a variety, i.e. in particular reduced, then any constructible $\alpha \in X_\infty$ is *weakly stable* in the sense of [DL99a].

Theorem 3.2.13 ([dFD20, Theorem 10.8])**.** *Let X be a scheme of finite type over a perfect field k. For every $\alpha \in X_\infty$, we have*

$$\operatorname{edim}(\mathcal{O}_{X_\infty,\alpha}) < \infty$$

if and only if α is a constructible point and not contained in $(\operatorname{Sing} X)_\infty$.

The proof of Theorem 3.2.13 relies on a variant of the Birational Transformation Rule established in [DL99a]. We refer the reader to [dFD20, Section 9] for more details.

3.2.3 The formal neighborhood of a degenerate arc

Let X be a scheme locally of finite type over a field k and X_∞ the arc space of X. In this section we are interested in those arcs $\alpha \in X(k)$ which are contained in the complement of the smooth locus X_{sm} of X; that is, $\alpha \in (X \setminus X_{\mathrm{sm}})_\infty$. In accordance with the literature we refer to such arcs α as *degenerate*; they should be thought of as "the most singular" points of X_∞. In fact, this idea will be made precise with the results of this section and Section 3.3.

For now let X be a scheme (not necessarily of finite type) over a field k. Let $\pi_{\infty,0} : X_\infty \to X$ be the canonical projection given by $\alpha \mapsto \alpha_0$. Then $\pi_{\infty,0}$ has a section $\tau_{\infty,0} : X \to X_\infty$ given as follows: for each k-algebra R we map $x \in X(R)$ to the arc given by

$$\operatorname{colim}_n \operatorname{Spec}(R) \times_k \operatorname{Spec}(k[[t]]_n) \xrightarrow{\mathrm{pr}_1} \operatorname{Spec}(R) \xrightarrow{x} X.$$

If $X = \operatorname{Spec}(A)$ is affine, then $\tau_{\infty,0}$ corresponds to the ring map $A_n = \mathbb{HS}_k^\infty(A) \mapsto A$ given by

$$a^{(i)} \mapsto \begin{cases} a, & i = 0 \\ 0, & i > 0. \end{cases}$$

For $x \in X$ we call $\bar{x}_\infty := \tau_{\infty,0}(x)$ the *constant arc* centered at x. Clearly $\bar{x}_\infty$ degenerate is equivalent to $x \in X \setminus X_{\mathrm{sm}}$.

Similarly, for $n \in \mathbb{N}$ we obtain a section $\tau_{n,0} : X \to X_n$ to the projection $\pi_{n,0} : X_n \to X$ and we call $\bar{x}_n := \tau_{n,0}(x)$ the *constant n-jet* centered at x.

This section's main result describes the formal neighborhood of degenerate constant arcs as follows. Recall that a formal k-scheme $\mathcal{X}$ has a smooth factor if there exists a decomposition $\mathcal{X} \simeq \mathcal{Y} \times_k \operatorname{Spf}(k[[t]])$.

Theorem 3.2.14 ([CH17])**.** *Let X be a scheme over a field k of characteristic 0. Let $x \in X(k)$ and for $n \in \mathbb{N} \cup \{\infty\}$ let $\bar{x}_n$ be the corresponding constant arc resp. n-jet. Then the formal neighborhood $\widehat{(X_n)}_{\bar{x}_n}$ has a smooth factor if and only if $\widehat{X}_x$ has one.*

The idea of the proof is to combine the Zariski-Lipman-Nagata criterion (see Theorem 2.3.4) with the formula for the differentials in Theorem 2.2.15. Let us start by proving the easy direction first. We have the following lemma:

Lemma 3.2.15. *Let X be a scheme over k and $\alpha \in X_n(k)$. Assume that $\widehat{X}_{\alpha_0} \simeq \widehat{Y}_y \times \widehat{Z}_z$ for k-schemes Y, Z. Let $(\alpha_1, \alpha_2) \in (Y \times Z)_n(k)$ be the arc resp. n-jet corresponding to α. Then*

$$\widehat{(X_n)}_\alpha \simeq \widehat{(Y_n)}_{\alpha_1} \times \widehat{(Z_n)}_{\alpha_2}.$$

Proof. Follows from the description in Lemma 3.1.7. □

Corollary 3.2.16. *Let X be a scheme over k and $\alpha \in X_n(k)$. Assume the formal neighborhood $\widehat{X}_{\alpha_0}$ has a smooth factor, then so has $\widehat{(X_n)}_\alpha$.*

Proof of Theorem 3.2.14. We may assume that $X = \mathrm{Spec}(A)$ is affine. Write $A_n = \mathbb{HS}^n_k(A)$ and $\pi^* : A \to A_n$ for the map corresponding to $\pi_{n,0} : X_n \to X$ and $\tau^* : A_n \to A$ for the map corresponding to $\tau_{n,0} : X \to X_n$. If $\mathfrak{m} \subset A$ is the maximal ideal corresponding to $x \in X(k)$ and $\mathfrak{m}_n = \mathfrak{m}A_n$ its extension, then the identity on $A_\mathfrak{m}$ factors as

$$A_\mathfrak{m} \xrightarrow{\pi^*} (A_n)_{\mathfrak{m}_n} \xrightarrow{\tau^*} A_\mathfrak{m}.$$

Let us write $B = A_\mathfrak{m}$ and $B_n = (A_n)_{\mathfrak{m}_n}$. Then the factorization above lifts to the respective completions $\widehat{B}$ and $\widehat{B_n}$. Note also that, if $\gamma_n^* : B \to B_n[[t]]_n$ denotes the map corresponding to the universal arc resp. n-jet, then composing γ_n^* with the t-linear extension of τ^* yields the canonical inclusion $B \to B[[t]]_n$.

By Theorem 2.2.15 we have $\Omega_{B_n/k} \simeq \mathbb{HS}^n_{B/k}(\Omega_{B/k})$. Thus, for any module M over B_n we have isomorphisms

$$\begin{aligned}\mathrm{Der}_k(B_n, M) &\simeq \mathrm{Hom}_{B_n}(\Omega_{B_n/k}, M)\\ &\simeq \mathrm{Hom}_B(\Omega_B, M[[t]]_n)\\ &\simeq \mathrm{Der}_k(B, M[[t]]_n).\end{aligned}$$

Evaluating this isomorphism at $M = \widehat{B_n}$ and composing each derivation on the right hand side with $\widehat{\tau^*} : \widehat{B_n} \to \widehat{B}$ we obtain a map

$$\mathrm{Der}_k^{\mathrm{cont}}(\widehat{B_n}, \widehat{B_n}) \simeq \mathrm{Der}_k(B_n, \widehat{B_n}) \to \mathrm{Der}_k(B, \widehat{B}[[t]]_n) \simeq \mathrm{Der}_k^{\mathrm{cont}}(\widehat{B}, \widehat{B}[[t]]_n).$$

By the above remark, the $\widehat{B}$-module structure on $\widehat{B}[[t]]_n$ is given by the canonical inclusion $\widehat{B} \to \widehat{B}[[t]]_n$. We claim that, for any k-algebra R, we have $\mathrm{Der}_k(R, R[[t]]_n) \simeq \prod^n \mathrm{Der}_k(R, R)$. Indeed, let $d : R \to R[[t]]_n$ be a derivation and write $d(x) = \sum_{i=0}^n d_i(x)t^i$. Then

$$\sum_i d_i(xy)t^i = d(xy) = x\sum_{i=0}^n d_i(y)t^i + y\sum_{i=0}^n d_i(x)t^i = \sum_{i=0}^n (xd_i(y) + yd_i(x))t^i$$

and thus $d_i : R \to R$ is a k-derivation. Note that if $R = (R, \mathfrak{n})$ is local and d is regular (i.e. $d \mod \mathfrak{n} \not\equiv 0$), then there exists an i such that d_i is regular.

To summarize, we have a map

$$\mathrm{Der}_k^{\mathrm{cont}}(\widehat{B_n}, \widehat{B_n}) \to \prod^n \mathrm{Der}_k^{\mathrm{cont}}(\widehat{B}, \widehat{B}),$$

which takes a derivation d to a family of derivations $(d_i)_i$. Assume now that $\widehat{(X_n)}_{\bar{x}_n}$ has a smooth factor. By Theorem 2.3.4 this implies that there exists a regular derivation $d \in \mathrm{Der}_k^{\mathrm{cont}}(\widehat{B_n}, \widehat{B_n})$. Considering the image $(d_i)_i$ of d under the above map we see that there exists a regular derivation $d_i \in \mathrm{Der}_k^{\mathrm{cont}}(\widehat{B}, \widehat{B})$. Using Theorem 2.3.4 again in the other direction shows that $\widehat{X}_x$ has a smooth factor. □

Remark 3.2.17. The argument of Theorem 3.2.14 does not immediately extend to the case of positive characteristic, as there does not exist an identification similar to

$$\mathrm{Der}_k(R, R[[t]]_n) \simeq \prod^n \mathrm{Der}_k(R, R)$$

for higher derivations.

Remark 3.2.18. In [BS17c] a first example of a constant degenerate arc which does not have a DGK decomposition was given. The original motivation behind Theorem 3.2.14 in [CH17] was to prove that, in fact, all constant degenerate arcs do not allow such a decomposition. This statement will be generalized further in Section 3.4 to characterize general degenerate arcs as exactly those arcs not having a DGK decomposition.

3.3 Generic projections and embedding codimension of arc spaces

In Section 2.5 we have introduced two different notions of embedding codimension as a very coarse measure of the "size" of a singularity. The main result of this section is a characterization of those arcs who have finite embedding codimension as *non-degenerate* arcs, that is, those arcs which are not contained in the singular locus of X. In that case we will also provide an explicit bound for the embedding codimension. The first key result is Theorem 3.3.1, whose basic idea is that the cotangent map at a non-degenerate arc with respect to a general linear projection to an affine space is an isomorphism. This gives a geometric way of computing embedding codimension by considering the map of jet spaces at each level corresponding to the chosen linear projection. The aforementioned results then follow by the formulas for the embedding dimension of jets established in Section 3.2.2.

Throughout this section let X be a scheme of finite type over a field k. As in Section 3.1, given an arc $\alpha\colon \mathrm{Spec}(k[[t]]) \to X$, we will denote by α_0 and α_η the images in X of the closed point and the generic point of $\mathrm{Spec}(k[[t]])$; we call α_0 the *special point* of α and α_η the *the generic point* of α.

Given an open set $U \subset X$, we have $\alpha_\eta \in U$ if and only if the morphism $\alpha\colon \mathrm{Spec}(k[[t]]) \to X$ does not factor through the complement $X \setminus U$. We will be interested in the case where $U = X_{\mathrm{sm}}$, the smooth locus of X. Note that if k is perfect, then the complement $X \setminus X_{\mathrm{sm}}$ is the singular locus $\mathrm{Sing}(X)$ of X.

The following theorem is a variant of [dFD20, Theorems 9.2 and 9.3]. Moreover, the argument employed is similar to the one used to prove [EM09, Theorem 1.1, Theorem 1.2].

Theorem 3.3.1. *Suppose that X is an affine scheme over a perfect field k. Let $\alpha \in X_\infty$ be an arc and let $d := \dim_{k(\alpha_\eta)}(\Omega_{X/k} \otimes_k k(\alpha_\eta))$. Assume that one of the following holds:*

1. *k is a field of characteristic zero, or*
2. *$\alpha \in X_\infty(k)$.*

Fix a closed embedding $X \subset \mathbb{A}^N$, let $f\colon X \to Y := \mathbb{A}^d$ be the morphism induced by a general linear projection $\mathbb{A}^N \to \mathbb{A}^d$, and let $\beta := f_\infty(\alpha) \in Y_\infty$. Let $\mathfrak{m} \subset$

$\mathcal{O}_{X_\infty,\alpha}$ *and* $\mathfrak{n} \subset \mathcal{O}_{Y_\infty,\beta}$ *be the respective maximal ideals and* L *and* L' *the residue fields. Then the induced cotangent map*

$$(T_\alpha f_\infty)^* \colon \mathfrak{n}/\mathfrak{n}^2 \otimes_{L'} L \to \mathfrak{m}/\mathfrak{m}^2$$

is an isomorphism.

Proof. By assumption, we have that $\mathrm{ord}_\alpha(\mathrm{Fitt}^d(\Omega_{X/k})) < \infty$ and by taking a general linear projection we can ensure that

$$\mathrm{ord}_\alpha(\mathrm{Fitt}^d(\Omega_{X/k})) = \mathrm{ord}_\alpha(\mathrm{Fitt}^0(\Omega_{X/Y})).$$

Since k is perfect, we have a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{n}/\mathfrak{n}^2 \otimes_{L'} L \arrow[r] \arrow[d, "(T_\alpha f_\infty)^*"] & \Omega_{Y_\infty/k} \otimes_{\mathcal{O}_{Y_\infty}} L \arrow[r] \arrow[d, "\varphi"] & \Omega_{L'/k} \otimes_{L'} L \arrow[r] \arrow[d, "\delta"] & 0 \\
0 \arrow[r] & \mathfrak{m}/\mathfrak{m}^2 \arrow[r] & \Omega_{X_\infty/k} \otimes_{\mathcal{O}_{X_\infty}} L \arrow[r] & \Omega_{L/k} \arrow[r] & 0
\end{tikzcd}$$

The main step is to understand the map φ. As in the proof of [dFD20, Theorems 9.2], denote for short $B_L := L[[t]]$ and $Q_L := \rho_*(\mathcal{Q}_\infty) \otimes_{\mathcal{O}_{X_\infty}} L$. Note that, writing $X = \mathrm{Spec}(A)$, we have that $Q_L \simeq Q_\infty \otimes_{A_\infty} L$, where Q_∞ is defined as in Section 2.2.

Note that, by Theorem 3.2.7, there are natural isomorphisms

$$\Omega_{X_\infty/k} \otimes_{\mathcal{O}_{X_\infty}} L \simeq \Omega_{X/k} \otimes_{\mathcal{O}_X} Q_L$$

and

$$\Omega_{Y_\infty/k} \otimes_{\mathcal{O}_{Y_\infty}} L \simeq \Omega_{Y/k} \otimes_{\mathcal{O}_Y} Q_L.$$

We will use these isomorphisms to study φ.

By pulling back the terms of the exact sequence

$$\Omega_{Y/k} \otimes_{O_Y} \mathcal{O}_X \to \Omega_{X/k} \to \Omega_{X/Y} \to 0$$

along α, we obtain the exact sequence

$$\Omega_{Y/k} \otimes_{\mathcal{O}_Y} B_L \to \Omega_{X/k} \otimes_{\mathcal{O}_X} B_L \to \Omega_{X/Y} \otimes_{\mathcal{O}_X} B_L \to 0.$$

Since Y is smooth, we see that the term $F_Y := \Omega_{Y/k} \otimes_{\mathcal{O}_Y} B_L$ is a free B_L-module. Write $\Omega_{X/k} \otimes_{\mathcal{O}_X} B_L = F_X \oplus T_X$ where F_X is free and T_X is torsion. Since $\mathrm{ord}_\alpha(\mathrm{Fitt}^0(\Omega_{X/Y})) < \infty$, the term $T_{X/Y} := \Omega_{X/Y} \otimes_{\mathcal{O}_X} B_L$ is a torsion B_L-module, and we get an exact sequence:

$$0 \to F_Y \to F_X \oplus T_X \to T_{X/Y} \to 0.$$

Since Q_L is a divisible Q_L-module, tensoring with Q_L kills torsion, and hence the above sequence gives the exact sequence

$$0 \to \mathrm{Tor}_1^{B_L}(T_X, Q_L) \to \mathrm{Tor}_1^{B_L}(T_{X/Y}, Q_L) \to F_Y \otimes_{B_L} Q_L \xrightarrow{\varphi'} F_X \otimes_{B_L} Q_L \to 0.$$

Note that $\varphi' = \varphi$ under the aforementioned isomorphisms. We have $\mathrm{Tor}_1^{B_L}(T_X, Q_L) \simeq T_X$, and this has dimension $\mathrm{ord}_\alpha(\mathrm{Fitt}^d(\Omega_{X/k}))$ over L. Similarly, $\mathrm{Tor}_1^{B_L}(T_{X/Y}, Q_L) \simeq T_{X/Y}$ has dimension $\mathrm{ord}_\alpha(\mathrm{Fitt}^0(\Omega_{X/Y}))$ over L. Since these two dimensions are equal, the map φ' in the sequence above is an isomorphism. We conclude that φ is an isomorphism.

The surjectivity of φ implies that δ is surjective, and the injectivity of δ follows from our assumption that either (1) or (2) holds. We conclude that $(T_\alpha f_\infty)^*$ is an isomorphism. $\square$

Corollary 3.3.2. *Keeping the assumptions and notation from Theorem 3.3.1, let α_n, β_n denote the images of α, β under the projections $X_\infty \to X_n$ and $Y_\infty \to Y_n$, and $\mathfrak{m}_n \subset \mathcal{O}_{X_n,\alpha_n}$, $\mathfrak{n}_n \subset \mathcal{O}_{Y_n,\beta_n}$ denote the corresponding ideals with residue fields L_n, L'_n. Then the induced cotangent map*

$$(T_{\alpha_n} f_n)^* \colon \mathfrak{n}_n/\mathfrak{n}_n^2 \otimes_{L'_n} L \to \mathfrak{m}_n/\mathfrak{m}_n^2 \otimes_{L_n} L$$

is injective for all $n \in \mathbb{N}$.

Proof. This follows from the diagram

$$\begin{tikzcd}
\mathfrak{n}/\mathfrak{n}^2 \otimes_{L'} L \arrow[r] & \mathfrak{m}/\mathfrak{m}^2 \\
\mathfrak{n}_n/\mathfrak{n}_n^2 \otimes_{L'_n} L \arrow[r] \arrow[u] & \mathfrak{m}_n/\mathfrak{m}_n^2 \otimes_{L_n} L \arrow[u]
\end{tikzcd}$$

and the fact that the top horizontal and left vertical arrows are injections. $\square$

Theorem 3.3.3. *Let X be a scheme of finite type over a perfect field k and $\alpha \in X_\infty$. Assume that one of the following holds:*

1. *k is a field of characteristic zero, or*
2. *$\alpha \in X_\infty(k)$.*

Then we have

$$\operatorname{ecodim}(\mathcal{O}_{X_\infty,\alpha}) \le \limsup_{n\to\infty} \operatorname{ecodim}\big(\mathcal{O}_{X_n,\alpha_n}\big)$$

where α_n is the image of α under the truncation map $\pi_n \colon X_\infty \to X_n$.

Proof. We can assume without loss of generality that X is affine. Given a map

$$f \colon X \to Y := \mathbb{A}^d,$$

we let $\beta := f_\infty(\alpha) \in Y_\infty$. For every n, we denote by $\alpha_n \in X_n$ and $\beta_n \in Y_n$ the images of α and β at the respective n-jet schemes. For ease of notation, we also set $A_\infty := \mathcal{O}_{X_\infty,\alpha}$ and $B_\infty := \mathcal{O}_{Y_\infty,\beta}$ and, for every $n \in \mathbb{N}$, $A_n := \mathcal{O}_{X_n,\alpha_n}$ and $B_n := \mathcal{O}_{Y_n,\beta_n}$. For every $n \in \mathbb{N} \cup \{\infty\}$, we denote by $\mathfrak{m}_n \subset A_n$ and $\mathfrak{n}_n \subset B_n$ the respective maximal ideals, and by $L_n := A_n/\mathfrak{m}_n$ and $L'_n := B_n/\mathfrak{n}_n$ the residue fields.

Note that we have direct systems $\{A_n \to A_{n+1} \mid n \in \mathbb{N}\}$ and $\{B_n \subset B_{n+1} \mid n \in \mathbb{N}\}$, and $A_\infty = \operatorname{colim}_n A_n$ and $B_\infty = \operatorname{colim}_n B_n$. Moreover, we have commutative diagrams

$$\begin{tikzcd}
B_\infty \arrow[r, "\varphi"] & A_\infty \\
B_n \arrow[r, "\varphi_n"] \arrow[u, hook] & A_n \arrow[u]
\end{tikzcd}$$

where $\mathfrak{n} = \varphi^{-1}(\mathfrak{m})$, $\mathfrak{n}_n = \varphi_n^{-1}(\mathfrak{m}_n)$, $\mathfrak{n}_n = \mathfrak{n} \cap B_n$, and $\mathfrak{m}_n = \mathfrak{m} \cap A_n$.

For every $n \in \mathbb{N} \cap \{\infty\}$, let

$$D\varphi_n \colon \mathfrak{n}_n/\mathfrak{n}_n^2 \otimes_{L'_n} L_n \to \mathfrak{m}_n/\mathfrak{m}_n^2$$

be the induced cotangent map.

We pick f as in Theorem 3.3.1. For every $n \in \mathbb{N} \cup \{\infty\}$, there is an associated map of graded rings $\mathrm{gr}(\varphi_n) \colon \mathrm{gr}(B_n) \to \mathrm{gr}(A_n)$. We denote by

$$\psi_n \colon \mathrm{gr}(B_n) \otimes_{L'_n} L_\infty \to \mathrm{gr}(A_n) \otimes_{L_n} L_\infty.$$

the map induced by $\mathrm{gr}(\varphi_n)$ by the indicated base changes.

Note that $\mathfrak{m}_\infty = \mathrm{colim}_n\, \mathfrak{m}_n$ and hence $\mathfrak{m}_\infty^r = \mathrm{colim}_n\, \mathfrak{m}_n^r$ for all r. Indeed, if $a \in \mathfrak{m}_\infty^r$ for some $r \geq 2$, then we can write $a = a_1 \dots a_r$ with $a_i \in \mathfrak{m}$; then we can pick n such that such that $a_i \in \mathfrak{m}_n$ for all i and hence $a \in \mathfrak{m}_n^r$. It follows that

$$\mathrm{gr}(A_\infty) = \operatorname*{colim}_n \mathrm{gr}(A_n) \otimes_{L_n} L_\infty,$$

and similarly we have

$$\mathrm{gr}(B_\infty) = \operatorname*{colim}_n \mathrm{gr}(B_n) \otimes_{L'_n} L'_\infty.$$

Since $\mathfrak{n}_n^r = \varphi_n^{-1}(\mathfrak{m}_n^r)$ for all r, for every n we have a commutative diagram

$$\begin{tikzcd}
\mathrm{gr}(B_\infty) \otimes_{L'_\infty} L_\infty \arrow[r, "\psi_\infty"] & \mathrm{gr}(A_\infty) \\
\mathrm{gr}(B_n) \otimes_{L'_n} L_\infty \arrow[r, "\psi_n"] \arrow[u, hook] & \mathrm{gr}(A_n) \otimes_{L_n} L_\infty \arrow[u]
\end{tikzcd} .$$

For short, let $R_n := \mathrm{gr}(A_n) \otimes_{L_n} L_\infty$, $S_n := \mathrm{gr}(B_n) \otimes_{L'_n} L_\infty$, and $K_n := \ker(\psi_n)$.

Lemma 3.3.4. $\mathrm{ht}(K_\infty) = \limsup_n \mathrm{ht}(K_n)$.

Proof. First, note that $K_\infty = \mathrm{colim}_n\, K_n$. Indeed, the inclusion $K_\infty \supset \mathrm{colim}_n\, K_n$ is clear, and conversely, if $b \in K_\infty$ and we fix $n \in \mathbb{N}$ such that $b \in S_n$, then $\psi_m(b)$ is in the kernel of $R_m \to R_\infty$ for all $m \geq n$ and hence, since each $\psi_m(b)$ maps to $\psi_{m+1}(b)$ via $R_m \to R_{m+1}$, it follows that $\psi_m(b)$ is zero for $m \gg n$, which means that $b \in K_m$ for $m \gg n$.

We are now ready to prove that

$$\mathrm{ht}(K_\infty) = \limsup_n \mathrm{ht}(K_n).$$

Note that the maps $S_n \to S_\infty$ are extensions of polynomial rings over the same field L_∞. Thus they are faithfully flat and hence for every prime $\mathfrak{p}_n \subset S_n$ its extension $\mathfrak{p}_n S_\infty$ is prime.

Consider first the case where $\mathrm{ht}(K_\infty) < \infty$ and let $\mathfrak{p} \subset S_\infty$ be a minimal prime over K_∞ with $\mathrm{ht}(\mathfrak{p}) = \mathrm{ht}(K_\infty)$. By Remark 1.6.11 we have that $\mathfrak{p}$ is finitely generated by elements $f_1, \dots, f_r \in S_\infty$. For each $n > 0$ let $\mathfrak{p}_n$ be any minimal prime over K_n contained in $\mathfrak{p} \cap S_n$. Then $\mathfrak{p}' := \mathrm{colim}_n\, \mathfrak{p}_n$ is a prime of S_∞ with $K_\infty \subset \mathfrak{p}' \subset \mathfrak{p}$, so $\mathfrak{p}' = \mathfrak{p}$. Let $n_1 > 0$ be such that $f_1, \dots, f_r \in \mathfrak{p}_{n_1}$, then $\mathfrak{p}_n = \mathfrak{p} \cap S_n$ for $n \geq n_1$. Given any chain of primes

$$(0) = \mathfrak{q}_0 \subsetneq \mathfrak{q}_1 \subsetneq \cdots \subsetneq \mathfrak{q}_t = \mathfrak{p} \subset S_\infty,$$

pick $s_i \in \mathfrak{q}_i \setminus \mathfrak{q}_{i-1}$, and fix n_2 such that $s_1, \dots, s_t \in S_{n_2}$. Then for every $n \geq \max\{n_1, n_2\}$ we get a chain of primes

$$(0) = \mathfrak{q}_0 \cap S_n \subsetneq \mathfrak{q}_1 \cap S_n \subsetneq \cdots \subsetneq \mathfrak{q}_t \cap S_n = \mathfrak{p}_n.$$

Thus $\operatorname{ht}(K_\infty) \leq \limsup_n \operatorname{ht}(K_n)$. The other inequality follows by the going-down theorem applied to $S_n \to S_\infty$.

If $\operatorname{ht}(K_\infty) = \infty$, then, since $\operatorname{ht}(K_\infty) \geq \operatorname{ht}(K_n S_\infty)$, a similar argument shows that the sequence $\{\operatorname{ht}(K_n)\}_n$ is unbounded. □

We can now finish the proof of the theorem. Since B_n is formally smooth, for every $n \in \mathbb{N} \cup \{\infty\}$ the natural map

$$\operatorname{Sym}_{L'_n}(\mathfrak{n}_n/\mathfrak{n}_n^2) \to \operatorname{gr}(B_n)$$

is an isomorphism (see Remark 2.5.14 for the case $n = \infty$). Furthermore, the diagram

$$\begin{array}{ccc} S_n & \xrightarrow{\psi_n} & R_n \\ {\scriptstyle\simeq}\uparrow & & \uparrow{\scriptstyle\gamma_n} \\ \operatorname{Sym}_{L'_n}(\mathfrak{n}_n/\mathfrak{n}_n^2) \otimes_{L'_n} L_\infty & \xrightarrow{\sigma_n} & \operatorname{Sym}_{L_n}(\mathfrak{m}_n/\mathfrak{m}_n^2) \otimes_{L_n} L_\infty \end{array}$$

is commutative.

By Theorem 3.3.1, the map σ_∞ is an isomorphism, and hence

$$\operatorname{ht}(K_\infty) = \operatorname{ht}(\ker(\gamma_\infty)) = \operatorname{ecodim}(A_\infty).$$

Similarly, by Corollary 3.3.2 σ_n is an injective L-linear map of polynomial rings, and we have

$$\operatorname{ht}(K_n) \leq \operatorname{ht}(\ker(\gamma_n)) = \operatorname{ecodim}(A_n).$$

Then we conclude by Lemma 3.3.4. □

Theorem 3.3.5. *Let X be a scheme of finite type over a perfect field k and $\alpha \in X_\infty$. Assume that either k is a field of characteristic zero, or α is a k-rational point. Then we have*

$$\operatorname{ecodim}(\mathcal{O}_{X_\infty,\alpha}) \leq \operatorname{ord}_\alpha(\operatorname{Fitt}^d(\Omega_{X/k}))$$

where $d = \dim_{\alpha_\eta}(X)$. In particular:

1. *If X is a variety then* $\operatorname{ecodim}(\mathcal{O}_{X_\infty,\alpha}) \leq \operatorname{ord}_\alpha(\operatorname{Jac}_X)$.
2. *If $\alpha_\eta \in X_{\mathrm{sm}}$ and $X^0 \subset X$ is the irreducible component containing α_η, then* $\operatorname{ecodim}(\mathcal{O}_{X_\infty,\alpha}) \leq \operatorname{ord}_\alpha(\operatorname{Jac}_{X^0}) < \infty$.

Proof. First note that it suffices to prove the theorem when $\alpha_\eta \in X_{\mathrm{sm}}$, as otherwise the right hand side of the stated inequality is infinite and the statement is trivial. Let us therefore assume that $\alpha_\eta \in X_{\mathrm{sm}}$.

For every r, let $J_r := \mathrm{Fitt}^r(\Omega_{X/k}) \subset \mathcal{O}_X$. On the one hand, for every finite n we have by Lemma 3.2.10 that

$$\mathrm{edim}(\mathcal{O}_{X_n,\alpha_n}) = (n+1)d_n - \dim(\overline{\{\alpha_n\}}) + \mathrm{ord}_\alpha(J_{d_n}),$$

where $d_n = d(\alpha_n, \Omega_{X/k})$ is the Betti number of $\Omega_{X/k}$ with respect to α_n (see Section 3.2.2) and $\overline{\{\alpha_n\}}$ the closure of α_n in X_n. On the other hand, since α is not in an irreducible component of X_∞ that is fully contained in $(\mathrm{Sing}(X))_\infty$, we have

$$\dim(\mathcal{O}_{X_n,\alpha_n}) \geq (n+1)d - \dim(\overline{\{\alpha_n\}})$$

for all finite n. Since for all n large enough we have $d_n = d$, we deduce by Proposition 2.5.2 that $\mathrm{ecodim}(\mathcal{O}_{X_n,\alpha_n}) \leq \mathrm{ord}_\alpha(J_d)$ for all $n \gg 1$. We conclude by Theorem 3.3.3 that $\mathrm{ecodim}(\mathcal{O}_{X_\infty,\alpha}) \leq \mathrm{ord}_\alpha(J_d)$, as stated.

Regarding the last two assertions of the theorem, (1) follows by the fact that if X is a variety then, by definition, $\mathrm{Jac}_X = \mathrm{Fitt}^d(\Omega_{X/k})$. As for (2), if $\alpha_\eta \in X_{\mathrm{sm}}$ then by Lemma 3.3.6 we have $\widehat{\mathcal{O}_{X_\infty,\alpha}} \simeq \widehat{\mathcal{O}_{X^0_\infty,\alpha}}$, and hence we can apply (1) to X^0; note also that in this case we have $\alpha \in X^0_{\mathrm{sm}}$ and hence $\mathrm{ord}_\alpha(\mathrm{Jac}_{X^0}) < \infty$. □

We include a proof of the following property, which is well known to experts and is remarked in [Dri02].

Lemma 3.3.6. *Let X be a scheme of finite type over a field k and $\alpha \in X_\infty$ an arc with $\alpha_\eta \in X_{\mathrm{sm}}$. Let $X^0 \subset X$ be the irreducible component containing α_η. Then $\widehat{\mathcal{O}_{X_\infty,\alpha}} \simeq \widehat{\mathcal{O}_{X^0_\infty,\alpha}}$.*

Proof. We may assume that $X = \mathrm{Spec}(R)$ is affine. By abuse of notation we write α for the map $R \to A[[t]]$. Let $\mathfrak{a} := \ker(\alpha)$. If $(0) = \prod_i \mathfrak{q}_i \subset R$ is a primary decomposition with $\mathfrak{q}_0$ the minimal prime defining X^0, then the condition $\alpha_\eta \in X^0$ translates to $\mathfrak{q}_0 \subset \mathfrak{a}$ and $\mathfrak{q}_i \not\subset \mathfrak{a}$ for $i \neq 0$. Let A be a test-ring, i.e., A is local with maximal ideal $\mathfrak{m}$, residue field K equal to the residue field of $\alpha \in X_\infty$, and $\mathfrak{m}^n = 0$ for some $n \in \mathbb{N}$. Let α' be any A-deformation of α, that is, given by a map $R \to A[[t]]$. To prove the lemma, it suffices to show that $\mathfrak{a}' := \ker(\alpha') \supset \mathfrak{q}_0$. We have the commutative diagram

$$\begin{tikzcd} R \arrow[r, "\alpha'"] \arrow[dr, "\gamma'"'] & A[[t]] \arrow[r] \arrow[d] & L[[t]] \arrow[d] \\ & A((t)) \arrow[r] & L((t)) \end{tikzcd}$$

where $A(\!(t)\!)$ denotes the localization of $A[[t]]$ at the ideal $\mathfrak{m}$. Since $A[[t]] \to A(\!(t)\!)$ is injective, we have $\mathfrak{a}' = \ker(\gamma')$. Let $f \in \mathfrak{q}_0$. Take any $f_i \in \mathfrak{q}_i \setminus \mathfrak{a}$ for $i \neq 0$. Then $g := f \prod_i f_i \in \mathfrak{a}'$. Since $\gamma'(f_i) \neq 0$ modulo $\mathfrak{m}$, we have that $\gamma'(f_i)$ is a unit. Thus $0 = \gamma'(g) = \gamma'(f)u$ where u is a unit, and in particular $f \in \mathfrak{a}'$. □

Theorem 3.3.7. *Let X be a scheme of finite type over a perfect field k. For $\alpha \in X_\infty$ such that $\alpha_\eta \in X \setminus X_{\mathrm{sm}}$, we have* $\mathrm{ecodim}(\mathcal{O}_{X_\infty,\alpha}) = \infty$.

Proof. Note that since k is perfect we have $X \setminus X_{\mathrm{sm}} = \mathrm{Sing}(X)$, and in particular the condition that $\alpha_\eta \in X \setminus X_{\mathrm{sm}}$ is equivalent to having $\alpha \in (\mathrm{Sing}(X))_\infty$.

For every $n \in \mathbb{N}$, let $\pi_n \colon X_\infty \to X_n$ be the truncation morphism, and let $\alpha_n := \pi_n(\alpha) \in X_n$. Note that for $n = 0$ this notation is consistent with the one already introduced for α_0. Let L and L_n denote the residue fields of X_∞ and α and of X_n at α_n. By Lemma 3.2.11 (see also Remark 3.2.9), for all n sufficiently large the cotangent map

$$(T_\alpha \pi_n)^* \colon \mathfrak{m}_{\alpha_n}/\mathfrak{m}_{\alpha_n}^2 \otimes_{L_n} L \to \mathfrak{m}_\alpha/\mathfrak{m}_\alpha^2$$

has rank at least $(n+1)d(\alpha) - \dim(\overline{\{\alpha_n\}})$, where

$$d(\alpha) := \dim_{k(\alpha(\eta))}(\Omega_{X/k} \otimes k(\alpha(\eta))).$$

Then, by Proposition 2.5.7, we have

$$\begin{aligned} \mathrm{ecodim}(\mathcal{O}_{X_\infty,\alpha}) &\geq (n+1)d(\alpha) - \mathrm{trdeg}_k(L_n) - \dim\left(\mathcal{O}_{\overline{\pi_n(X_\infty)},\alpha_n}\right) \\ &= (n+1)d(\alpha) - \dim_{\alpha_n}(\overline{\pi_n(X_\infty)}) \end{aligned}$$

where $\overline{\pi_n(X_\infty)}$ denotes the Zariski closure of $\pi_n(X_\infty)$ in X_n.

Since X is of finite type, X_∞ has finitely many irreducible components (see Corollary 3.1.17).This implies that for n sufficiently large we have

$$\dim_{\alpha_n}(\overline{\pi_n(X_\infty)}) = \max_{C \ni \alpha} \dim(\overline{\pi_n(C)})$$

where the maximum is taken over the irreducible components C of X_∞ that contain α.

Let C be one of the irreducible components of X_∞ containing α, let $\beta \in C$ be its generic point, and let $Z \subset X$ be the closure of β_η in X. From Lemma 3.2.12 we have

$$\dim(\overline{\pi_n(C)}) \leq (n+1)\dim(Z) \leq (n+1)\dim_{\alpha_0}(X).$$

Since $\alpha_\eta \in \mathrm{Sing}(X)$, we see by the definition of $d(\alpha)$ that $d(\alpha) > \dim_{\alpha_0}(X)$, and therefore

$$\lim_{n\to\infty} \big((n+1)d(\alpha) - \dim(\overline{\pi_n(C)})\big) \geq \lim_{n\to\infty} (n+1)\big(d(\alpha) - \dim_{\alpha_0}(X)\big) = \infty.$$

We conclude that $\mathrm{ecodim}(\mathcal{O}_{X_\infty,\alpha}) = \infty$, as claimed. ☐

Corollary 3.3.8. *Let X be a scheme of finite type over a field k and $\alpha \in X_\infty$. Assume that either k has characteristic zero, or $\alpha \in X_\infty(k)$. Then we have $\alpha_\eta \in X_{\mathrm{sm}}$ if and only if* $\mathrm{ecodim}(\mathcal{O}_{X_\infty,\alpha}) < \infty$.

Proof. If k has characteristic zero, then the corollary follows by Theorems 3.3.5 and 3.3.7.

Let then k be any field, and assume that $\alpha \in X_\infty(k)$. For a field extension $k \subset k'$, we denote $X' := X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k')$ and let $\alpha' \colon \mathrm{Spec}(k'[[t]]) \to X'$ be the arc obtained by base change from α. Since a point of X is in the smooth locus if and only if it is geometrically regular, we can find a field extension $k \subset k'$ such that α' is not a regular point of X'. By faithfully flat descent of regularity, we can replace k' with a larger field extension and assume without loss of generality that k' is perfect. Note that $X'_\infty \cong X_\infty \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k')$, and hence $\mathcal{O}_{X'_\infty,\alpha'} \simeq \mathcal{O}_{X_\infty,\alpha} \otimes_k k'$. Then, by Proposition 2.5.10, we have

$$\mathrm{ecodim}(\mathcal{O}_{X'_\infty,\alpha'}) = \mathrm{ecodim}(\mathcal{O}_{X_\infty,\alpha}).$$

This reduces to the case of perfect fields, where the result follows from again by Theorems 3.3.5 and 3.3.7. □

3.4 On the Drinfeld–Grinberg–Kazhdan theorem

Theorem 3.3.5 can be seen as a finiteness statement for singularities of the arc space at arcs that are not fully contained in the singular locus. One of the first major results in this direction is the theorem of Drinfeld, Grinberg, and Kazhdan, which we will state below in its version in [Dri02]. Recall that for any equicharacteristic local ring $(A, \mathfrak{m}, k)$ a DGK decomposition is an isomorphism $\widehat{A} \simeq k[[t_i \mid i \in I]]/\mathfrak{a}$, where $\mathfrak{a}$ is an ideal of finite polynomial definition.

Theorem 3.4.1 ([GK00, Theorem 2.1], [Dri02, Theorem 0.1])**.** *Let X be a scheme of finite type over a field k, and let $\alpha \in X_\infty(k)$. If $\alpha_\eta \in X_{\mathrm{sm}}$, then the local ring $\mathcal{O}_{X_\infty,\alpha}$ admits a DGK decomposition.*

As mentioned in Remark 1.6.3, any DGK decomposition of $\mathcal{O}_{X_\infty,\alpha}$ induces an isomorphism of formal schemes

$$\widehat{(X_\infty)_\alpha} \cong \widehat{Z}_z \times \Delta^{\mathbb{N}},$$

with Z a scheme of finite type over k, $z \in Z(k)$ and $\Delta = \mathrm{Spf}(k[[t]])$. As in Definition 2.3.10, we call the formal scheme $\widehat{Z}_z$ a formal model for α. If there exists a formal model for X, then, by the results on cancellation in Section 2.3.2, there also exists a minimal formal model $\widehat{Z}_z$, unique up to isomorphism, which has no smooth factors, that is, it is not of the form $\mathcal{Y} \times \Delta$. We denote a minimal formal model for α by $\mathcal{Z}^{\min}_\alpha$.

We shall remark here that the formal model provided by Theorem 3.4.1 is not minimal in general, as we will see in Section 3.5. While we will not provide a full proof of Theorem 3.4.1 here, at the end of this section we will briefly mention how the Weierstrass preparation theorem is used as an essential ingredient in the proof.

Combining Theorem 3.4.1 with the results of Section 3.3, we obtain the following characterization of k-rational arcs admitting a DGK decomposition. The result also gives an explicit bound for the embedding codimension; we should stress that such bound does not follow from the proofs in [GK00; Dri02; Dri18].

Theorem 3.4.2. *Let X be a scheme of finite type over a field k. For any $\alpha \in X_\infty(k)$, the following are equivalent:*

1. *$\alpha_\eta \in X_{\mathrm{sm}}$.*
2. *$\mathcal{O}_{X_\infty,\alpha}$ admits a DGK decomposition.*
3. *$\mathcal{O}_{X_\infty,\alpha}$ admits a weak DGK decomposition.*

4. $\operatorname{ecodim}(\mathcal{O}_{X_\infty,\alpha}) < \infty$.

Moreover, if k is perfect and $\alpha_\eta \in X_{\rm sm}$, then

$$\operatorname{ecodim}(\mathcal{O}_{X_\infty,\alpha}) \le \operatorname{ord}_\alpha(\operatorname{Jac}_{X^0})$$

where $X^0 \subset X$ is the irreducible component containing α_η.

Proof. The implication (1) $\Rightarrow$ (2) is Theorem 3.4.1, the implication (2) $\Rightarrow$ (3) is obvious, the implication (3) $\Rightarrow$ (4) follows from Corollary 1.6.10, and finally Corollary 3.3.8 gives the implication (4) $\Rightarrow$ (1). The last statement follows from Theorem 3.3.5. ☐

Example 3.4.3. Let X be the hypersurface defined by $x_0x_{n+1}+f(x_1,\dots,x_n)=0$ and $\alpha \in X(k)$ the arc given by $(t,0,\dots,0) \in k[[t]]^{n+2}$. Assume further that the hypersurface $H \subset \mathbb{A}^n$ given by $f(x_1,\dots,x_n)=0$ has a singularity at 0. Then, as shown in [Dri02], a DGK decomposition for $\mathcal{O}_{X_\infty,\alpha}$ is given by

$$\widehat{\mathcal{O}_{X_\infty,\alpha}} \simeq k[[x_1,\dots,x_n]]/f(x_1,\dots,x_n) \mathbin{\widehat{\otimes}}_k k[[t_i \mid i \in \mathbb{N}]].$$

The singularity of α is thus again given by H and $\operatorname{ecodim}(\mathcal{O}_{X_\infty,\alpha}) = 1$. On the other hand, the order of α with respect to the Jacobian ideal Jac_X is 1, hence the bound in Theorem 3.4.2 is sharp in this case.

Example 3.4.4. Similarly as in the previous example, let X be the hypersurface defined by $x_0x_{n+1}+f(x_1,\dots,x_n)=0$ where f is a polynomial of multiplicity 2, and take this time $\alpha \in X_\infty(k)$ to be the arc given by $(t^m,0,\dots,0) \in k[[t]]^{n+2}$. Denoting by $g^{(j)}$ the j-th higher derivative of an element $g \in k[x_0,\dots,x_{n+1}]$ and setting for short $I=\{0,1,\dots,n+1\}$ and $J=\mathbb{Z}_{\ge 0}$, X_∞ is defined by the ideal $\mathfrak{a} = ((x_0x_{n+1}+f)^{(j)} \mid j \in J)$ of $P := k[x_i^{(j)} \mid (i,j) \in I \times J]$. Let $\mathfrak{m} \subset P$ be the maximal ideal at α. Since $x_0^{(m)}$ is a unit in the local ring $P_\mathfrak{m}$, we see that the ideal $\operatorname{in}(\mathfrak{a}P_\mathfrak{m})$ is generated by the elements $x_{n+1}^{(j)}$ for $j \in J$, and $\operatorname{in}(f^{(l)})$ for $0 \le l \le m-1$. As long as f is chosen so that $\operatorname{in}(f^{(l)})$, for $0 \le l \le m-1$, form a regular sequence (e.g., $f = x_1x_2$ would work), we get that $\mathcal{O}_{X_\infty,\alpha}$ has embedding codimension m. Since clearly the order of α with respect to Jac_X is also m, this shows that the bound in Theorem 3.4.2 is sharp for all possible orders of the arc with the Jacobian ideal of X.

Let us mention here the following consequence of Theorem 3.4.2, which implies that the local rings of closed arcs in the arc space provide plenty of examples of non-Noetherian rings for which their embedding codimension agrees with their formal embedding codimension.

Corollary 3.4.5. *If X is a scheme of finite type over a field k, then*

$$\operatorname{ecodim}(\mathcal{O}_{X_\infty,\alpha}) = \operatorname{fcodim}(\mathcal{O}_{X_\infty,\alpha})$$

for every $\alpha \in X_\infty(k)$. If k is perfect, then the same holds for all constructible points $\alpha \in X_\infty$ with $\alpha_\eta \in X_{\rm sm}$.

Proof. Assume first that $\alpha \in X_\infty(k)$. If $\alpha_\eta \in X_{\rm sm}$ then the equality follows by Theorem 3.4.1 and Proposition 2.5.17. If $\alpha_\eta \in X \setminus X_{\rm sm}$, then we have $\operatorname{ecodim}(\mathcal{O}_{X_\infty,\alpha}) = \infty$ by Corollary 3.3.8, and we conclude by Theorem 2.5.19. Suppose now that $\alpha \in X_\infty$ is a constructible point with $\alpha_\eta \in X_{\rm sm}$. By Theorem 3.2.13, $\mathcal{O}_{X_\infty,\alpha}$ has finite embedding dimension, and hence the assertion follows by Corollary 2.5.16. ☐

We now state the following application of Theorems 3.3.1 and 3.4.1, which says that a generic projection of the base scheme induces an efficient DGK decomposition at an arc that is not contained in the singular locus. The proof relies on Theorem 3.4.7, which we will prove afterwards.

Theorem 3.4.6. *Let $X \subset \mathbb{A}^N$ be an affine scheme of finite type over a perfect field k, $\alpha \in X_\infty(k)$ with $\alpha_\eta \in X_{\rm sm}$ and $d = \dim_{\alpha_\eta}(X)$. Let $f \colon X \to Y := \mathbb{A}^d$ be the map induced by a general linear projection $\mathbb{A}^N \to \mathbb{A}^d$, and let $\beta := f_\infty(\alpha)$. Then the associated map*

$$\varphi \colon \mathcal{O}_{Y_\infty,\beta} \to \mathcal{O}_{X_\infty,\alpha},$$

gives an efficient formal embedding of $\mathcal{O}_{X_\infty,\alpha}$. Moreover, if k is infinite, then there exist formal coordinates $u_i \in \widehat{\mathcal{O}_{Y_\infty,\beta}}$, $i \in \mathbb{N}$, such that $\ker(\widehat{\varphi})$ is generated by finitely many polynomials in u_i, hence $\widehat{\varphi}$ induces an efficient DGK decomposition.

Proof. The first part follows from Theorem 3.3.1 together with the fact that $\mathcal{O}_{Y_\infty,\beta}$ is formally smooth over k. Regarding the second assertion, we know by Theorem 3.4.1 that the map φ induces a surjection

$$\psi \colon \widehat{\mathcal{O}_{Y_\infty,\beta}} \to \widehat{\mathcal{O}_{Z,z}} \,\widehat{\otimes}\,_k k[[t_i \mid i \in \mathbb{N}]]$$

with Z a scheme of finite type over k and $z \in Z(k)$. If Z is smooth at z then there is nothing to show. Otherwise, by Theorem 3.4.7, we may assume that $Z \subset \mathbb{A}^n$, where $n = \operatorname{edim}(\mathcal{O}_{Z,z})$. Since ψ induces an isomorphism of continuous cotangent spaces the statement follows from Theorem 1.3.17. $\square$

The following result is well-known in the case of complex varieties (see for example [BK84, Theorem 3]) and we provide an extension of the proof to arbitrary fields of infinite cardinality.

Theorem 3.4.7. *Let X be a scheme of finite type over an infinite field k and $x \in X(k)$. If $\operatorname{edim}(X, x) = d$ and X is not smooth at x, then there exists a closed subscheme $Y \subset \mathbb{A}^d_k$, a point $y \in Y(k)$, and an isomorphism*

$$\mathcal{O}_{Y,y} \simeq \mathcal{O}_{X,x}.$$

Proof. We may assume that X is projective and embedded in $\mathbb{P}^n$ for some $n > d$. Denote by $\bar{k}$ the algebraic closure of k and write $\bar{X} := X \times_k \operatorname{Spec}(\bar{k})$ and $\bar{x}$ for the $\bar{k}$-point on $\bar{X}$ corresponding to x. As $\mathcal{O}_{\bar{X},\bar{x}}$ is not a regular ring, we have

$$\dim_{\bar{x}}(\bar{X}) < \operatorname{edim}(\bar{X}, \bar{x}) = \operatorname{edim}(X, x) = d.$$

Suppose we can find a linear projection $\pi \colon \mathbb{P}^n \to \mathbb{P}^d$ defined over k such that, if $\bar{Y}$ denotes the scheme-theoretic image of $\bar{X}$ under π and $\bar{y} = \pi(\bar{x})$, then the induced map $\mathcal{O}_{\bar{Y},\bar{y}} \to \mathcal{O}_{\bar{X},\bar{x}}$ is an isomorphism. Since $\bar{Y} = Y \times_k \operatorname{Spec}(\bar{k})$, where Y is the scheme-theoretic image of X under the linear projection centered at x, we obtain a map $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ whose base change to $\bar{k}$ gives the map above. Thus, by faithfully flat descent, we get that $\mathcal{O}_{Y,y} \simeq \mathcal{O}_{X,x}$.

Now, in order to prove the claim, let $\bar{T} \subset \mathbb{P}^n_{\bar{k}}$ be the unique linear space passing through $\bar{x}$ whose tangent space at $\bar{x}$ agrees with that of $\bar{X}$. Furthermore, let $\bar{S}$ be the closure of the set of all lines connecting $\bar{z}$ with $\bar{x}$, where $\bar{z} \in \bar{X}$,

$\bar{z} \neq \bar{x}$. Note that $\dim(\bar{S}) = \dim(\bar{X}) + 1 \leq d$. Consider now the closure $\bar{Z}$ of the set $\bar{X} \cup \bar{T} \cup \bar{S}$, equipped with its reduced scheme structure. Since $\dim(\bar{S}) \leq d$ the set of all linear spaces $\bar{L}$ with $\bar{L} \cap \bar{Z} = \emptyset$ is open inside $\mathrm{Gr}(n-d-1, n) \times_k \mathrm{Spec}(\bar{k})$. The preimage of this set in $\mathrm{Gr}(n-d-1, n)$ is a nonempty open set, and since k is infinite, it has a k-rational point, which we denote by L. Hence we have that the corresponding projection $\pi_L \colon \mathbb{P}^n \to \mathbb{P}^d$, defined over k, satisfies $\pi_{\bar{L}}^{-1}(\bar{y}) \cap \bar{X} = \{\bar{x}\}$ set theoretically, where $\bar{y}$ corresponds to the k-point $y := \pi_L(x)$. Writing $\bar{Y} := \pi_{\bar{L}}(\bar{X})$ we get that the map of local rings $\mathcal{O}_{\bar{Y}, \bar{y}} \to \mathcal{O}_{\bar{X}, \bar{x}}$ is injective and finite. Since $\bar{L} \cap \bar{T} = \emptyset$ the tangent spaces of $\bar{x}$ and $\bar{y}$ are isomorphic and thus $\mathfrak{m}_{\bar{y}} \mathcal{O}_{\bar{X}, \bar{x}} = \mathfrak{m}_{\bar{x}}$. The claim now follows from the Nakayama lemma. $\square$

Let us conclude this section with next an example that illustrates in concrete terms the content of Theorem 3.4.6 when X is a hypersurface in an affine space. In this case the existence of the efficient formal embedding as in theorem can be verified directly from the equations.

Example 3.4.8. Let $f \in k[x_1, \dots, x_n, y]$ and X be the hypersurface defined by f. For the sake of convenience, we will write $x = (x_1, \dots, x_n)$. Let $\alpha = (x(t), y(t))$ be an arc on X such that $\mathrm{ord}_\alpha(\mathrm{Jac}_X) = \mathrm{ord}_t(\frac{\partial f}{\partial y}(x(t), y(t))) = d > 0$. We write $x(t) = \sum_j a^{(j)} t^j$ and $y(t) = \sum_j b^{(j)} t^j$; note that $a^{(j)} = (a_1^{(j)}, \dots, a_n^{(j)})$. Let $D = (D_p)_{p \geq 0}$ be the universal Hasse–Schmidt derivation on $k[x^{(j)}, y^{(j)} \mid j \geq 0]$, where $x^{(j)} = (x_1^{(j)}, \dots, x_n^{(j)})$. Then $X_\infty = \mathrm{Spec}(R_\infty)$, where

$$R_\infty = k[x^{(j)}, y^{(j)} \mid j \geq 0]/(f^{(p)} \mid p \geq 0),$$

with $f^{(p)} := D_p(f)$. Note that $f^{(p)}$ depends only on $x^{(j)}, y^{(j)}$ for $j \leq p$. The arc α then corresponds to the ideal $\mathfrak{m}_\alpha$ of R_∞ given by

$$\mathfrak{m}_\alpha = (x^{(j)} - a^{(j)}, y^{(j)} - b^{(j)} \mid j \geq 0).$$

Setting $\widetilde{f^{(p)}}(x^{(j)}, y^{(j)}) := f^{(p)}(x^{(j)} + a^{(j)}, y^{(j)} + b^{(j)})$, we get that

$$\mathcal{O}_{X_\infty, \alpha} \simeq k[x^{(j)}, y^{(j)}]_{(x^{(j)}, y^{(j)})}/(\widetilde{f^{(p)}}).$$

We are going to make use of the following explicit formula from [dFD20, Section 5]:

$$\frac{\partial f^{(p)}}{\partial y^{(q)}} = D_{p-q}\left(\frac{\partial f}{\partial y}\right), \quad q \leq p.$$

The condition $\mathrm{ord}_t(\frac{\partial f}{\partial y}(x(t), y(t))) = d$ implies that, for $p \geq d$,

$$\frac{\partial \widetilde{f^{(p)}}}{\partial y^{(q)}}(0,0) = \frac{\partial f^{(p)}}{\partial y^{(q)}}(a, b) = D_{p-q}\left(\frac{\partial f}{\partial y}\right)(a, b) \begin{cases} = 0, & p - d < q \leq p \\ \neq 0, & q = p - d. \end{cases}$$

Now the above implies that the initial forms of $f^{(p)}$, for $p \geq d$, can be written as

$$\mathrm{in}(\widetilde{f^{(d+i)}}) = y^{(i)} + g^{(d+i)}$$

where $g^{(d)} \in k[x^{(j)} \mid j \leq d]$ and, for $i > 0$, $g^{(d+i)} \in k[x^{(j)}, y^{(l)} \mid j \leq d + i, l < i]$. In particular, the elements $x^{(j)}$ and $\widetilde{f^{(d+j)}}$, for $j \geq 0$, give formal coordinates in $k[[x^{(j)}, z^{(j)} \mid j \geq 0]]$, hence the map

$$\varphi \colon k[[x^{(j)}, z^{(j)} \mid j \geq 0]] \to k[[x^{(j)}, y^{(j)} \mid j \geq 0]], \quad x^{(j)} \mapsto x^{(j)},\ z^{(j)} \mapsto \widetilde{f^{(j+d)}}$$

is an isomorphism. Write $h_i := \varphi^{-1}(f^{(i)})$ and $\mathfrak{a} := (\bar{h}_0, \dots, \bar{h}_{d-1})$, where $\bar{h}_i$ is obtained from h_i by setting $z^{(j)} = 0$ for all $j \geq 0$. Then we get that

$$\widehat{\mathcal{O}_{X_\infty,\alpha}} \simeq k[[x^{(j)} \mid j \geq 0]]/\widehat{\mathfrak{a}}.$$

Observe that the map $k[[x^{(j)} \mid j \geq 0]] \to \widehat{\mathcal{O}_{X_\infty,\alpha}}$ is the efficient formal embedding from Theorem 3.4.6 with respect to the projection $(x, y) \mapsto x$. However, this isomorphism does not induce a priori a DGK decomposition a priori since the ideal $\widehat{\mathfrak{a}}$ is not necessarily of finite polynomial definition with respect to the variables $x^{(j)}$.

3.4.1 On Weierstrass preparation

Let us elaborate a bit on the proof of the Drinfeld–Grinberg–Kazhdan theorem and in particular, the use of Weierstrass preparation therein. First we recall the statement of Weierstrass preparation which we proved in Section 1.2.

Theorem 3.4.9 (Corollary 1.2.29)**.** *Let $(A, \mathfrak{m})$ be strongly admissible. Let $f \in A[[t]]$ with $\mathrm{ord}_t(\bar{f}) = s < 0$, where $\bar{f}$ is obtained from f by taking all coefficients module $\mathfrak{m}$. Then there exists a unique $u \in A[[t]]^*$ and a unique Weierstrass polynomial $q \in A[t]$ of degree s such that $f = uq$.*

Recall that a Weierstrass polynomial of degree s was a monic polynomial q of degree s in $A[t]$ such that $\bar{q} = t^s$. The assumption that A is complete is crucial here: consider the polynomial $f = t^2 + t + x$ in $k[x][[t]]$. The Weierstrass factorization of f in $k[[x, t]]$ is given by $(t - \alpha_1)(t - \alpha_2)$, where α_1 and α_2 are the roots of f in $k[[x]]$.

In fact, the fact that A is required to be complete for Weierstrass factorization to hold is one of the reasons why the Drinfeld-Grinberg-Kazhdan theorem does not extend beyond the formal neighborhood. We would like to demonstrate this via the slightly rephrased proof Drinfeld gave in [Dri18]. The key result (of which we will not give a proof) is the following proposition:

Proposition 3.4.10 ([Dri18, Proposition 3.1.2])**.** *Let X be a complete intersection in $\mathbb{A}_k^{n+m}$ and Δ_X the subscheme given by the vanishing of an $m \times m$-minor Q of the Jacobian matrix of X. Then there exists a scheme Z representing the following functor: to each k-algebra R we associate the set of pairs (q, γ), where $q \in R[t]$ is monic of degree N and $\gamma \in \lim_r X(R[t]/(q^r))$ such that the scheme-theoretic preimage of Δ_X in $\mathrm{Spec}(R[t]/(q^2)$ equals $\mathrm{Spec}(R[t]/(q))$. Moreover, we have $Z \simeq Y \times \mathbb{A}^I$, where Y is a scheme of finite type over k and $\mathbb{A}^I$ the affine space over some (typically infinite) set I.*

In fact, the scheme Z is constructed as a limit of schemes Z_r of finite type such that, for $r \geq 3$, the map $Z_{r+1} \to Z_r$ is a trivial affine fibration. In fact, the formal completion of the scheme Z_2 will give the formal model defined in the original proof in [Dri02]. We will investigate its structure further in Section 3.5.

For now, we want to derive Theorem 3.4.1 from the above proposition. As described in [Dri18], we may assume that X is a complete intersection and the non-degenerate arc $\alpha : \mathrm{Spec}(k[[t]]) \to X$ satisfies $\alpha^{-1}(\Delta_X) = \mathrm{Spec}(k[t]/(t^N))$. Setting $q = t^N$ we obtain $z = (t^N, \alpha) \in Z(k)$. Let us first define a morphism $\widehat{Z}_z \to \widehat{(X_\infty)}_\alpha$ of formal neighborhoods. To that avail, let $(A, \mathfrak{m})$ be a test-ring and $(t^N + \widetilde{q}, \widetilde{\alpha})$ an A-deformation of z (see Corollary 1.1.36). In particular, $t^N + \widetilde{q}$ is a monic polynomial of degree N such that $\widetilde{q} \in \mathfrak{m}[t]$. We make the following observation:

Lemma 3.4.11. *Let $(A, \mathfrak{m})$ be a test ring and $q \in A[t]$ a monic polynomial of degree N such that $q \equiv t^N$ modulo $\mathfrak{m}$. Then the (q)-adic topology on $A[[t]]$ is equivalent to the (t)-adic one.*

Proof. Write $q = t^N + \widetilde{q}$ with $\widetilde{q} \in \mathfrak{m}[t]_{<N}$. By assumption there exists $c > 0$ such that $\mathfrak{m}^c = 0$. Thus t divides q^c. On the other hand, considering the expansion

$$\frac{1}{1 + \frac{\widetilde{q}}{t^N}} = 1 + \frac{\widetilde{q}}{t^N} + \ldots + \left(\frac{\widetilde{q}}{t^N}\right)^{c-1}$$

gives an element $u \in A[[t]]$ such that $qu = t^{Nc}$. □

In fact, in [Ngo17] a *strict Weierstrass factorization* of $f \in R[[t]]$ (with R an arbitrary ring) was defined to be a Weierstrass factorization $f = uq$ with the additional property that q divides a power of t. This distinction was made to study such factorizations over non-complete rings, with the overarching goal of proving a more global variant of the Drinfeld–Grinberg–Kazhdan theorem. We refer the reader to [Ngo17] for more details.

Coming back to our proof, by the above lemma we see that the element $\widetilde{\alpha} \in \lim_r X(A[t]/(q^r))$ gives rise to a morphism $\operatorname{Spec}(A[[t]]) \to X$ which is easily seen to be a deformation of α. On the other hand, given a deformation $\widetilde{\alpha} : \operatorname{Spec}(A[[t]]) \to X$ let $q' = \widetilde{\alpha}^*(Q) \in A[[t]]$. By assumption $q' \equiv t^N$ modulo $\mathfrak{m}$ and thus we can apply Weierstrass preparation to find a Weierstrass factorization $q' = qu$. Then the pair $(q, \widetilde{a})$ defines a deformation of z in Z. The two assigments are easily seen to be inverse to each other. Thus Theorem 3.4.1 follows from the second assertion of the proposition.

One fact that deserves to be highlighted is that the isomorphism of formal neighborhoods constructed above (which is essentially the same as the one given in [Dri02]) does not arise as the completion of a morphism of schemes in either direction, since neither Weierstrass preparation nor Lemma 3.4.11 hold for more general rings. We will contrast this in the next section with the formal embedding obtained by considering a general linear projection as in Theorem 3.3.1 and Theorem 3.4.6.

3.5 Efficient embeddings of the formal model

Our aim this section is to compare the formal embedding given by Theorem 3.4.6 to the one provided by the Drinfeld–Grinberg–Kazhdan theorem, with the precise statement given as Theorem 3.5.2. We first need to recall the construction of the Drinfeld models.

Let $X \subset \mathbb{A}^N$ be an affine scheme of finite type over a field k, consider a k-rational arc $\alpha \in X_\infty(k)$ such that $\alpha_\eta \in X_{\mathrm{sm}}$, and let $d := \dim_{\alpha_\eta}(X)$ and $c := N - d$. Let X^0 be the irreducible component of X containing α_η (note that $d = \dim X^0$), and let $X' \supset X^0$ be the complete intersection scheme defined by the vanishing of c general linear combinations $p_1, \ldots, p_c$ of a set of generators of the ideal of X^0 in $\mathbb{A}^N$. As explained in [Dri02], the respective inclusions induce isomorphisms $\widehat{\mathcal{O}_{X_\infty,\alpha}} \simeq \widehat{\mathcal{O}_{X^0_\infty,\alpha}} \simeq \widehat{\mathcal{O}_{X'_\infty,\alpha}}$ (detailed proofs are given in Lemma 3.3.6 and [BS17b, Section 4.2]). Pick coordinates $x_1, \dots, x_d, y_1, \dots, y_c$ in the ambient affine space $\mathbb{A}^N$. For a general choice of such coordinates, we can assume that

$$\operatorname{ord}_\alpha\left(\det\left(\frac{\partial(p_1,\dots,p_c)}{\partial(y_1,\dots,y_c)}\right)\right) = \operatorname{ord}_\alpha(\mathrm{Jac}_{X'}) = \operatorname{ord}_\alpha(\mathrm{Jac}_{X^0}) =: e < \infty.$$

Drinfeld defines a specific formal model for $\widehat{\mathcal{O}_{X_\infty,\alpha}}$ depending only on the choices of the coordinates x_i, y_j, the equations p_l, and the order of contact e. Concretely, consider the affine space $\mathbb{A}^m$ where $m = e(1+2d+c)$. We denote by $R[t]_{<n}$ the space of polynomials of degree $< n$ with coefficients in R. Denoting by Q_n the scheme representing the functor $R \mapsto t^n + R[t]_{<n}$, the space of monic polynomials of degree n with coefficients in R, we identify $\mathbb{A}^m$ with the product $Q_e \times \mathbb{A}^d_{2e-1} \times \mathbb{A}^c_{e-1}$. Under this identification, a k-rational point of $\mathbb{A}^m$ corresponds to a triple

$$(q(t), \bar{x}(t), \bar{y}(t)) \in (t^e + k[t]_{<e}) \times (k[t]_{<2e})^d \times (k[t]_{<e})^c.$$

In particular, coordinates in $\mathbb{A}^m$ take the form $q^{(n)}, \bar{x}_i^{(n)}, \bar{y}_j^{(n)}$. Consider the conditions

$$\begin{aligned}
p_1(\bar{x}(t),\bar{y}(t)) \;\equiv\; \cdots \;\equiv\; p_c(\bar{x}(t),\bar{y}(t)) \quad &\equiv \quad 0 \qquad \mod q(t),\\
\det\left(\frac{\partial(p_1,\dots,p_c)}{\partial(y_1,\dots,y_c)}(\bar{x}(t),\bar{y}(t))\right) \quad &\equiv \quad 0 \qquad \mod q(t),\\
\operatorname{adj}\left(\frac{\partial(p_1,\dots,p_c)}{\partial(y_1,\dots,y_c)}(\bar{x}(t),\bar{y}(t))\right)\begin{pmatrix} p_1(\bar{x}(t),\bar{y}(t)) \\ \vdots \\ p_c(\bar{x}(t),\bar{y}(t))\end{pmatrix} \quad &\equiv \quad \begin{pmatrix} 0 \\ \vdots \\ 0\end{pmatrix} \qquad \mod q(t)^2.
\end{aligned} \tag{3.5a}$$

Here $\operatorname{adj}(B)$ denotes the classical adjoint of a matrix B. As explained in [Dri02] and [BS17b, Sections 3.3 and 3.4], the conditions in (3.5a) are polynomial in the coefficients of $q(t), \bar{x}(t), \bar{y}(t)$, and therefore they define a finite type subscheme $Z \subset \mathbb{A}^m$.

Write the arc α in the coordinates (x, y) of $\mathbb{A}^N$ as $\alpha = (a(t), b(t))$ where $a(t) \in k[[t]]^d$ and $b(t) \in k[[t]]^c$. To α we associate the point $z = (t^e, \bar{a}(t), \bar{b}(t)) \in Z$ given by

$$\bar{a}(t) \equiv a(t) \mod t^{2e}, \qquad \bar{b}(t) \equiv b(t) \mod t^e. \tag{3.5b}$$

It is shown in [Dri02] that $\widehat{Z}_z$ gives a (finite-dimensional) formal model for α, that is,

$$\widehat{(X_\infty)_\alpha} \simeq \widehat{Z}_z \times \Delta^{\mathbb{N}}. \tag{3.5c}$$

The isomorphism in (3.5c) can be expressed somewhat explicitly in coordinates. We identify $\mathbb{A}^d_\infty$ with an infinite-dimensional affine space $\mathbb{A}^{\mathbb{N}}$, and we use the notation $\xi(t)$ for points in $\mathbb{A}^{\mathbb{N}}$. Hence coordinates in $\mathbb{A}^{\mathbb{N}}$ take the form $\xi_i^{(n)}$, with $1 \le i \le d$ and $n \ge 0$. The disk $\Delta^{\mathbb{N}}$ appearing in (3.5c) is the formal neighborhood of $c(t)$ in $\mathbb{A}^{\mathbb{N}}$, where $c(t) := t^{-2e}(a(t))_{\ge 2e}$ is the truncation of $a(t)$ to degrees $\ge 2e$ divided by t^{2e}. Summarizing, we have described coordinates $(x(t), y(t))$ in $\widehat{(X_\infty)_\alpha}$ and coordinates $(q(t), \bar{x}(t), \bar{y}(t), \xi(t))$ in $\widehat{Z}_z \times \Delta^{\mathbb{N}}$. As explained in [Dri02], the isomorphism in (3.5c) gives the relation

$$x(t) = q(t)^2 \xi(t) + \bar{x}(t),$$

and we have

$$\bar{x}(t) \equiv x(t) \mod q(t)^2, \qquad \bar{y}(t) \equiv y(t) \mod q(t). \tag{3.5d}$$

We emphasize that these relations only hold at the level of formal neighborhoods.

Notice that the point $z \in Z$ depends on the arc α, but the scheme Z only depends on the choices of the coordinates x_i, y_j, the equations p_l, and the order of contact e. The choice of coordinates x_i, y_j also determines the linear projection $\mathbb{A}^N \to \mathbb{A}^d$ given by $(x, y) \mapsto x$, and hence the induced map $f\colon X \to \mathbb{A}^d$.

Definition 3.5.1. With the above notation, we say that (Z, z) is a *Drinfeld model* of X_∞ at α, and that it is *compatible* with f.

We are now ready to state and prove our comparison theorem.

Theorem 3.5.2. *Let $X \subset \mathbb{A}^N$ be an affine scheme of finite type over a perfect field k, let $\alpha \in X_\infty(k)$ with $\alpha_\eta \in X_{\mathrm{sm}}$, and let $d := \dim_{\alpha_\eta}(X)$. Let $f\colon X \to Y := \mathbb{A}^d$ be induced by a general linear projection $\mathbb{A}^N \to \mathbb{A}^d$ and $\beta := f_\infty(\alpha)$. Let (Z, z) be a Drinfeld model compatible with f, and let $\widehat{(X_\infty)_\alpha} \simeq \widehat{Z}_z \times \Delta^{\mathbb{N}}$ be the corresponding DGK decomposition.*

1. *The composition map*

$$\widehat{Z}_z \times \Delta^{\mathbb{N}} \xrightarrow{\sim} \widehat{(X_\infty)_\alpha} \hookrightarrow \widehat{(Y_\infty)_\beta}$$

is the completion of a morphism $g\colon Z \times \mathbb{A}^{\mathbb{N}} \to Y_\infty$.

2. *If $X^0 \subset X$ is the irreducible component of containing α_η and we set $e := \mathrm{ord}_\alpha(\mathrm{Jac}_{X^0})$, then the composition map*

$$\widehat{Z}_z \hookrightarrow \widehat{Z}_z \times \Delta^{\mathbb{N}} \xrightarrow{\sim} \widehat{(X_\infty)_\alpha} \hookrightarrow \widehat{(Y_\infty)_\beta} \twoheadrightarrow \widehat{(Y_{2e-1})_{\beta_{2e-1}}}$$

is an efficient formal embedding. Moreover, at the level of associated graded rings, we have that

$$\mathrm{gr}(\mathcal{O}_{Z,z}) = \mathrm{Im}(\mathrm{gr}(\mathcal{O}_{Y_{2e-1},\beta_{2e-1}}) \to \mathrm{gr}(\mathcal{O}_{X_\infty,\alpha})).$$

Proof. We use the notation introduced at the beginning of the section. In particular: we have coordinates (x, y) in $\mathbb{A}^N$ such that the projection $\mathbb{A}^N \to Y$ is given by $(x, y) \mapsto x$; we write $\alpha = (a(t), b(t))$, so $\beta = f_\infty(\alpha) = a(t)$; we have a space $\mathbb{A}^m$ with coordinates $(q(t), \bar{x}(t), \bar{y}(t))$, and $Z \subset \mathbb{A}^m$ is defined by the conditions in (3.5a); the point $z \in Z$ is given by $z = (t^e, \bar{a}(x), \bar{b}(x))$ as in (3.5b); the formal scheme $\widehat{Z_z} \hat{\times} \Delta^{\mathbb{N}}$ is contained in the completion of $\mathbb{A}^m \times \mathbb{A}^{\mathbb{N}}$ at $(z, c(t))$ where $c(t) = t^{-2e}(a(t))_{\geq 2e}$, and the coordinates in this affine space have the form $(q(t), \bar{x}(t), \bar{y}(t), \xi(t))$.

We define a map $\mathbb{A}^m \times \mathbb{A}^{\mathbb{N}} \to Y_\infty$ via

$$(q(t), \bar{x}(t), \bar{y}(t), \xi(t)) \;\mapsto\; q(t)^2 \xi(t) + \bar{x}(t),$$

and we let $g\colon Z \times \mathbb{A}^{\mathbb{N}} \to Y_\infty$ be the restriction. It is clear from the discussion surrounding (3.5d) that the completion of g gives the composition of $\widehat{Z}_z \times \Delta^{\mathbb{N}} \xrightarrow{\sim} \widehat{(X_\infty)}_\alpha \hookrightarrow \widehat{(Y_\infty)}_\beta$, and the first statement of the theorem follows.

We compute the tangent map of g explicitly. With a small abuse of notation where coordinates of elements and coordinate functions are written in the same way, we denote a tangent vector on $Z \times \mathbb{A}^{\mathbb{N}}$ based at a point $(z, c(t)) = (t^e, \bar{a}(t), \bar{b}(t), c(t))$ by

$$(t^e + dq(t)\epsilon,\ \bar{x}(t) + d\bar{x}(t)\epsilon,\ \bar{y}(t) + d\bar{y}(t)\epsilon,\ c(t) + d\xi(t)\epsilon)$$

where $dq(t) \in k[t]_{<e}$, $d\bar{x}(t) \in (k[t]_{<2e})^d$, $d\bar{y}(t) \in (k[t]_{<e})^c$, $d\xi(t) \in k[t]^d$, and $\epsilon^2 = 0$. The image of such a tangent vector under g is given by

$$\begin{aligned}
&\big(t^e + dq(t)\epsilon\big)^2\big(c(t) + d\xi(t)\epsilon\big) + \big(\bar{x}(t) + d\bar{x}(t)\epsilon\big)\\
&\qquad = \big(t^{2e}c(t) + \bar{x}(t)\big) + \big(d\bar{x}(t) + d\xi(t)t^{2e} + 2c(t)dq(t)t^e\big)\epsilon\\
&\qquad = a(t) + \big(d\bar{x}(t) + d\xi(t)t^{2e} + 2c(t)dq(t)t^e\big)\epsilon.
\end{aligned}$$

In other words, the tangent map of g at $(z, c(t))$ is given by

$$dx(t) = d\bar{x}(t) + d\xi(t)t^{2e} + 2c(t)dq(t)t^e,$$

or, in coordinates (recall that $c(t) = t^{-2e}(a(t))_{\geq 2e}$), by

$$dx_i^{(n)} = \begin{cases} d\bar{x}_i^{(n)} & \text{if } n < e,\\ d\bar{x}_i^{(n)} + 2\sum_{k+l=n-e} a_i^{(k+2e)}dq^{(l)} & \text{if } e \leq n < 2e,\\ d\xi_i^{(n-2e)} + 2\sum_{k+l=n-e} a_i^{(k+2e)}dq^{(l)} & \text{if } n \geq 2e.\end{cases}$$

From this we see that g induces a surjective map between associated graded rings

$$\varphi\colon k\Big[\, dx_i^{(n)} \,\Big|\, {\textstyle{n\in\mathbb{N} \atop 1\leq i\leq d}} \Big] \simeq \operatorname{gr}(\mathcal{O}_{Y_\infty,\beta}) \longrightarrow \operatorname{gr}(\mathcal{O}_{Z,z}) \otimes_k k\Big[\, d\xi_i^{(m)} \,\Big|\, {\textstyle{m\in\mathbb{N} \atop 1\leq i\leq d}} \Big].$$

We have a commutative diagram

$$\begin{array}{ccc}
k\Big[\, dx_i^{(n)} \,\Big|\, {\textstyle{n\in\mathbb{N} \atop 1\leq i\leq d}} \Big] & \xrightarrow{\qquad\varphi\qquad} & \operatorname{gr}(\mathcal{O}_{Z,z}) \otimes_k k\Big[\, d\xi_i^{(m)} \,\Big|\, {\textstyle{m\in\mathbb{N} \atop 1\leq i\leq d}} \Big]\\
\Big\| & & \Big\downarrow \lambda\\
k\Big[\, dx_i^{(n)} \,\Big|\, {\textstyle{n<2e \atop 1\leq i\leq d}} \Big] \otimes_k k\Big[\, dx_i^{(n)} \,\Big|\, {\textstyle{n\geq 2e \atop 1\leq i\leq d}} \Big] & \xrightarrow{\psi\otimes_k\mu} & \operatorname{gr}(\mathcal{O}_{Z,z}) \otimes_k k\Big[\, d\xi_i^{(m)} \,\Big|\, {\textstyle{m\in\mathbb{N} \atop 1\leq i\leq d}} \Big]
\end{array}$$

where λ is the $\operatorname{gr}(\mathcal{O}_{Z,z})$-linear map given by

$$d\xi_i^{(m)} \ \mapsto\ d\xi_i^{(m)} - 2\sum_{k+l=m-e} a_i^{(k+2e)}dq^{(l)},$$

μ is given by $dx_i^{(n)} \mapsto d\xi_i^{(n-2e)}$ for $n \geq 2e$, and ψ agrees with the map $\operatorname{gr}(\mathcal{O}_{Y_{2e-1},\beta_{2e-1}}) \to \operatorname{gr}(\mathcal{O}_{Z,z})$ induced by the composition

$$Z \to Z \times \{c(t)\} \hookrightarrow Z \times \mathbb{A}^{\mathbb{N}} \to Y_\infty \to Y_{2e-1},$$

which is given by

$$(\bar{x}(t), \bar{y}(t), q(t)) \;\mapsto\; \bar{x}(t) + t^{-2e}(a(t))_{\geq 2e}\, q(t)^2 \quad \mathrm{mod}\ t^{2e}.$$

The map λ is invertible and, by Theorem 3.3.1, the map φ is surjective, thus ψ is surjective as well. This implies that

$$\mathrm{gr}(\mathcal{O}_{Z,z}) = \mathrm{Im}(\mathrm{gr}(\mathcal{O}_{Y_{2e-1},\beta_{2e-1}}) \to \mathrm{gr}(\mathcal{O}_{X_\infty,\alpha})),$$

and hence the last assertion follows. For the first part of (2), the fact that ψ is surjective implies that the map induced on completions $\widehat{O_{Y_{2e-1},\beta_{2e-1}}} \to \widehat{\mathcal{O}_{Z,z}}$ is surjective as well. The fact that this is an efficient embedding follows from the injectivity of the corresponding tangent map. $\square$

3.6 Applications to birational geometry

In this section we want to mention some applications of the results established in the previous sections, in particular the characterization obtained in Theorem 3.4.2, to birational geometry. Jet and arc spaces have been used to obtain results on various invariants related to the log-canonical threshold, see for example [Mus02] and [Ish13]. An in-depth treatment of the study of such invariants and their applications would be beyond the scope of this thesis; we will instead provide the reader with references where appropriate.

Throughout this section, let X be a variety over a field k of characteristic zero. Given a prime divisor E on a normal birational model $f\colon Y \to X$, we define the *Mather discrepancy* $\widehat{k}_E := \mathrm{ord}_E(\mathrm{Jac}_f)$ and the *Mather–Jacobian discrepancy* (or simply *Jacobian discrepancy*) $k_E^{\mathrm{MJ}} := \widehat{k}_E - \mathrm{ord}_E(\mathrm{Jac}_X)$ of E over X. Note that these definitions only depend on the valuation ord_E defined by E and not by the particular model chosen. The definition extends to any divisorial valuation $v = q\,\mathrm{ord}_E$, where q is a positive integer, by setting $\widehat{k}_v := q\widehat{k}_E$ and $k_v^{\mathrm{MJ}} := qk_E^{\mathrm{MJ}}$. When X is smooth, both discrepancies agree with the usual discrepancy of E over X. We say that X is *MJ-terminal* if $k_E^{\mathrm{MJ}} > 0$ whenever E is exceptional over X. As proved in [Ish13; dFD14], this condition is equivalent to the condition that if $X \subset Y$ is a closed embedding with Y smooth and $c = \mathrm{codim}(X, Y)$, then for any closed subset $T \subsetneq X$ the pair (Y, cX) has minimal log discrepancy $\mathrm{mld}_T(Y, cX) > 1$. We refer to [dFEI08; Ish13; dFD14; EI15] for general studies related to these invariants.

In [dFD20], Theorem 3.2.7 was used to obtain the following result which provides a formula for the Mather discrepancy in terms of the embedding dimension of the arc space:

Theorem 3.6.1 ([dFD20, Theorem 11.4])**.** *Let X be a reduced scheme of finite type over a perfect field k. For every divisorial valuation $q\,\mathrm{ord}_E$ on X there exists a unique maximal divisorial arc $\alpha \in X_\infty$ with $\mathrm{ord}_\alpha = q\,\mathrm{ord}_E$. Moreover:*

$$\mathrm{edim}(\mathcal{O}_{X_\infty,\alpha}) = q(\widehat{k}_E + 1).$$

This theorem, together with the results of Section 3.3, yields a new proof of the following theorem of Mourtada and Reguera.

Theorem 3.6.2 ([MR18, Theorem 4.1])**.** *With the above notation, let $\alpha \in X_\infty$ be the maximal arc defining a given divisorial valuation $q\,\mathrm{ord}_E$ (i.e., such that $\mathrm{ord}_\alpha = q\,\mathrm{ord}_E$). Then $\dim(\widehat{\mathcal{O}_{X_\infty,\alpha}}) \geq q(k_E^{\mathrm{MJ}} + 1)$.*

Proof. We have $\mathrm{edim}(\mathcal{O}_{X_\infty,\alpha}) = q(\widehat{k}_E + 1)$ by Theorem 3.6.1, and Theorem 3.3.5 gives us $\mathrm{ecodim}(\mathcal{O}_{X_\infty,\alpha}) \leq \mathrm{ord}_\alpha(\mathrm{Jac}_X)$. It follows then by Proposition 2.5.15 and Corollary 2.5.16 that

$$\dim(\widehat{\mathcal{O}_{X_\infty,\alpha}}) = \mathrm{edim}(\mathcal{O}_{X_\infty,\alpha}) - \mathrm{ecodim}(\mathcal{O}_{X_\infty,\alpha}) \geq q(k_E^{\mathrm{MJ}} + 1).$$

□

Assume now that X is an affine toric variety. To fix notation, let T be an algebraic k-torus, $N := \mathrm{Hom}_k(\mathbb{G}_m, T)$, $M := \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$, $\sigma \subset N_{\mathbb{R}}$ a rational convex cone, and $X := \mathrm{Spec}\, k[\sigma^\vee \cap M]$. Note that every $v \in \sigma \cap N$ defines a T-invariant divisorial valuation on X.

In their recent article [BS19b], Bourqui and Sebag study DGK decompositions of X_∞ at arcs that are not fully contained in the T-invariant divisor of X. The focus is on the open set $X_\infty^\circ := X_\infty \setminus (X \setminus T)_\infty$. They prove that for any $\alpha \in X_\infty^\circ$, the local ring $\mathcal{O}_{X_\infty,\alpha}$ only depends on the associated valuation ord_α, and in particular so does the minimal formal model [BS19b, Corollary 3.3]. In particular, if we set

$$X_{\infty,v}^\circ := \{\alpha \in X_\infty^\circ \mid \mathrm{ord}_\alpha = v\},$$

then we can denote by $\mathcal{Z}_v^{\min}$ the minimal formal model of X_∞ at any arc $\alpha \in X_{\infty,v}^\circ$.

The next theorem is one of the main results of [BS19b]. A similar, more general property is proved for elements v satisfying a certain property called $\mathcal{P}_v$; we refer to the original source for the precise statement.

Theorem 3.6.3 ([BS19b, Corollary 6.4])**.** *With the above notation, if v is indecomposable in $\sigma \cap N$, then the associated minimal formal model $\mathcal{Z}_v^{\min}$ has $\dim(\mathcal{Z}_v^{\min}) = 0$ and $\mathrm{edim}(\mathcal{Z}_v^{\min}) = \widehat{k}_v$.*

Indecomposable elements $v \in \sigma \cap N$ are characterized by the property that their centers on any resolution of singularity of X are irreducible components of codimension 1 of the exceptional locus, see [BS19b, Theorem 2.7]. In the terminology of the Nash problem, these form a particular class of *essential valuations*. By combining the above theorem with our results, we obtain the following corollary.

Corollary 3.6.4. *Let $X = \mathrm{Spec}\, k[\sigma^\vee \cap M]$ be an affine toric variety.*

1. *For any indecomposable element $v \in \sigma \cap N$, we have $k_v^{\mathrm{MJ}} \leq 0$.*

2. *If X is singular and $\mathbb{Q}$-factorial, then X is not MJ-terminal.*

Proof. Part (1) follows immediately from Theorem 3.6.3 and Theorem 3.4.2, and (2) follows from (1) and the observation that if X is singular and $\mathbb{Q}$-factorial then $\sigma \cap N$ necessarily contains an exceptional indecomposable element. This is just because the exceptional locus of any resolution of singularity of a $\mathbb{Q}$-factorial variety has always pure codimension 1, and the set of essential (toric) valuations is nonempty if X is singular. □

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Appendix

Abstract deutsch

Die vorliegende Dissertation beschäftigt sich vordergründig mit der Beschreibung der lokalen Geometrie von Arc Spaces von algebraischen Varietäten. Für eine positiv-dimensionale algebraische Varietät X ist deren Arc Space X_∞ ein nicht-Noethersches Schema von unendlicher Dimension, welches, grob gesprochen, Keime von formalen Kurven auf X parametrisiert. Die zugrundeliegenden lokalen Ringe von X_∞ und deren Vervollständigung sind Hauptgegenstand des ersten Kapitels. Als Musterbeispiel sei hier der Ring der formalen Potenzreihen in unendlich vielen Variablen genannt. Die im Noetherschen Fall verwendete adische Topologie wird hier durch die sogenannte *quasi-adische* Topologie ersetzt, und eine Erweiterung des klassischen Struktursatzes von Cohen bewiesen. Da viele Resultate der klassischen kommutativen Algebra nur für Noethersche Ringe gelten, werden unter anderem die Flachheit der Vervollständigung, das Konzept von Standardbasen sowie Endlichkeitseigenschaften von gewissen Klassen von Idealen für diese allgemeinere Klasse von Ringen untersucht.

Im zweiten Kapitel ist das Hauptaugenmerk auf die Verwendung von Derivationen und Differentialen zur Beschreibung von Singularitäten gelegt. Dabei werden zuerst höhere Derivationen von graduierten Ringen sowie Ribenboims Begriff von höheren Derivationen von Moduln eingeführt. Via der symmetrischen Algebra werden diese beiden Konzepte verglichen und als Konsequenz ein konzeptioneller Beweis der Formel von de Fernex und Docampo für die Differentialgarbe des Arc Space erhalten. Höher Derivationen werden auch für eine Verallgemeinerung des Zariski–Lipman–Nagata-Kriteriums für die Existenz von glatten Faktoren der formalen Umgebung verwendet. Schlußendlich wird die Einbettungskodimension als formal-lokale Invariante für Singularitäten von nicht-Noetherschen Schemata auf zwei Arten definiert: einerseits via des assoziierten graduierten Rings, andererseits über die Vervollständigung. Beide Varianten verallgemeinern den klassischen Begriff der Einbettungskodimension (oder Regularitätsdefekts) von algebraischen Varietäten; es wird zudem bewiesen, dass im Allgemeinen die formale Einbettungskodimension die graduierte von oben beschränkt.

Das dritte Kapitel ist ganz der lokalen Geometrie von Arc Spaces gewidmet. Aus der oben erwähnten Formel von de Fernex und Docampo für die Differentialgarbe wird mittels des Zariski–Lipman–Nagata-Kriteriums gefolgert, dass die formale Umgebung eines konstanten Arcs nur dann einen glatten Faktor besitzt, falls dies für die formale Umgebung des Basispunktes gilt. Als Hauptwerkzeug wird dann die Einbettungskodimension von Arcs verwendet;

mittels generischen linearen Projektionen wird gezeigt, dass diese endlich ist, wenn der Arc in einem glatten Punkt zentriert ist. In diesem Fall wird auch eine explizite Schranke für die Einbettungskodimension bewiesen. Dies liefert einerseits eine Charakterisierung von nicht-degenerierten Arcs mittels deren Einbettungskodimension; und andererseits eine Einbettung des formalen Modells wie im Satz von Drinfeld–Grinberg–Kazhdan in einen endlich-dimensionalen Jet Space. Abschließend werden noch als Anwendung zwei Resultate für die Mather–Jacobi-Diskrepanz von algebraischen Varietäten bewiesen.

Abstract englisch

The main objective of this thesis is to describe the local geometry of arc spaces of algebraic varieties. For a positive-dimensional algebraic variety X its arc space X_∞ is an infinite-dimensional scheme which, roughly speaking, parametrizes germs of formal curves on X. The local rings of the scheme X_∞ and their completions are the main objects of interest in the first chapter, with the prime example being the ring of infinite-variate formal power series over a field. Here it is necessary to replace the usual adic topology with what we call the *quasi-adic topology*, which will lead us to prove an extension of the Cohen strucure theorem for complete local rings. As many results of classical commutative algebra are usually only shown for Noetherian rings, several properties such as flatness of completion, existence of standard bases and various finiteness properties of ideals are investigated in this more general setting.

In the second chapter the main focus is put on the use of derivations and differentials to describe the algebraic and formal structure of singularities. Both higher derivations of graded rings and Ribenboim's notion of higher derivations of modules are introduced and compared using the symmetric algebra construction. As a particular consequence we obtain a conceptual proof of the formula of de Fernex and Docampo on the sheaf of differentials of the arc space. Higher derivations are also used in generalizing the Zariski–Lipman–Nagata criterion on the existence of smooth factors of the formal neighborhood. Finally we provide two extensions of the embedding codimension (or regularity defect) to non-Noetherian schemes, one using the associated graded and one using the formal completion. This provides a formal-local invariant which, heuristically speaking, is measuring the size of singularities. We also prove that, in general, the formal embedding codimension bounds its graded counterpart from above.

The third and final chapter is dedicated to the study of the local geometry of arc spaces. Using both the formula of de Fernex and Docampo and the Zariski–Lipman–Nagata criterion we prove that the formal neighborhood of a constant arc has a smooth factor if and only if the same is true for the formal neighborhood of its base point. Then we make use of generic linear projections to prove that the embedding codimension of an arc is finite if and only if the arc is centered in a smooth point. In this case an explicit bound for the embedding codimension is proven. On the one hand this gives a characterization of non-degenerate arcs via their embedding codimension; on the other hand we obtain an embedding of the formal model as in the Drinfeld–Grinberg–Kazhdan theorem into a finite-dimensional jet space. To finish, we provide two applications to the theory of Mather–Jacobi discrepancies of algebraic varieties.

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Conferences organized

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Peer reviewed publications

[C3] Higher derivations of modules and the Hasse-Schmidt module (with L Narváez-Macarro), 2020, to appear in Mich. Math. J., 13 pp.

Non-peer reviewed publications (recently submitted)

[C2] Embedding codimension of the space of arcs (with T. de Fernex and R. Docampo), 2020, submitted, 37 pp.
[C1] On the formal neighborhood of degenerate arcs (with H. Hauser), Preprint 2017, 9 pp.